On the Optimality of Decode and Forward for Some Cooperative Broadcast Channels

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***Abstract*—This article characterizes new boundary points on the capacity region of certain classes of more capable broadcast channels (BC) with uni-directional cooperation from the stronger to the weaker receiver. The new boundary points are achieved by a simple coding scheme that employs superposition coding at the transmitter with decode and forward at the stronger receiver. We evaluate our general result for Gaussian BCs and for a BC consisting of a binary erasure channel (BEC) to the stronger receiver and a binary symmetric channel (BSC) to the weaker receiver.**

***Index Terms*—Cooperative broadcast channels, degraded and more capable channels, decode and forward.**

I. Introduction

For non-cooperative broadcast channels (BC), the capacity region has been characterized for various classes of BCs. In particular, for BCs that are physically degraded (PD), stochastically degraded (SD), less noisy (LN), more capable (MC), essentially LN and MC [1], [2], and for the BCs with degraded message sets (asymmetric BCs, or ABCs, in the terminology of [3]). In all these BCs, superposition coding can achieve all rate-pairs in the capacity region.

For *cooperative* broadcast channels (CBC) [4]–[7], capacity results seem even more challenging. Nevertheless, the capacity region has been characterized for PD CBCs [5] and for one-sided cooperation in general ABCs [8], [9]. One sided-cooperation in semi-deterministic channels and cooperation for perfectly correlated Gaussian BCs were studied in [6]. Recently, a partial characterization of the capacity region of strongly less noisy channels with cooperation from the stronger user to the weaker was obtained in [7].

In most of these examples, capacity is exhausted by a simple scheme where the transmitter employs superposition coding (SPC) and the stronger (cooperative) user applies a decode and forward (D&F) [10] strategy so as to be able to send information to the weaker user that is only related to its desired message.

Our work is closely related to [7], which shows that above SPC and D&F strategy achieves a range of boundary points on the capacity region for certain classes of SD BCs with one-sided cooperation from the stronger to the weaker user and when the difference between the marginal channel strengths of the two users is large compared to the cooperation rate. In particular, [7] introduces a new *quantitative* notion of degradedness that applies to the CBC.

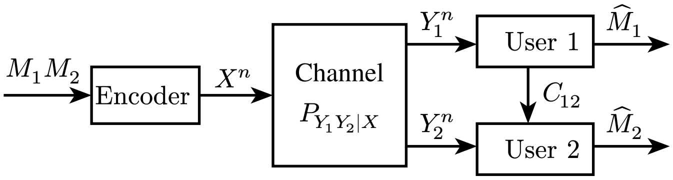

Fig. 1. Cooperative Broadcast Channel.

In this work, we improve on the results in [7]. That means, we consider MC CBCs with cooperation from the stronger user (User 1) to the weaker user (User 2), see Fig. 1, and we show that for some channels all rate-pairs (R_1, R_2) in the cooperative capacity region with rate R_1 below a given threshold can be achieved using SPC and D&F. The same conclusion was already obtained in [7] for a larger set of CBCs but for a smaller threshold on the rate R_1 and under more stringent constraints on the cooperation rate C_{12}.

Specifically, for a BC with a binary erasure channel (BEC) to the stronger user and a binary symmetric channel (BSC) to the weaker user and for parameters so that the BC is MC, we provide a larger set of capacity boundary points where SPC and D&F is optimal compared to the results obtained from [7, Thm. 1]. The improved result in [7, Thm. 4] was not evaluated for this example, but leads to the same result as in this work. Direct evaluation of [7, Thm. 4] however seems more cumbersome than the proof in this paper.

The current work also presents boundary points for the Gaussian BC where SPC and D&F is optimal. The previous result in [7, Thm. 1] fails to provide optimality of this scheme for any boundary point on the Gaussian CBC. The improved result [7, Thm. 4] was again not evaluated for the Gaussian case.

We also present a general result for optimality of SPC and D&F, beyond Gaussian BCs and BCs consisting of a BEC and BSC. Our general result holds for all MC CBCs where the SPC and D&F inner bound and the outer bound provided in [7] have similar parametric expressions, the bounds differing only in an additional sum-rate constraint for the inner bound. In our proof, we follow the idea in [7] and establish optimality of the SPC and D&F inner bound whenever the sum-rate constraint is not active. Unlike [7], which employs a statistical approach to show that the sum-rate constraint is not active, in this work, we directly exploit the parametric characterization.

The manuscript is organized as follows. The problem setup is introduced in Section II and the main results are presented in Section III. Proofs are provided in Section IV.

II. Channel Model and Preliminaries

A two-receiver broadcast channel $P_{Y_1,Y_2|X}$ consists of one input over a given alphabet $\mathcal{X}$ and two outputs over respective alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Under uni-directional cooperation, a link from User 1 to User 2, with capacity C_{12}, allows the users to cooperate.

For a given blocklength n, the encoder picks for each user $k = 1, 2$ a uniform message M_k over the set $\mathcal{N}_k = \{1, ..., \nu_k\}$. User 1 picks a cooperation message L_c from the set $\mathcal{N}_c = \{1, ..., 2^{nC_{12}}\}$ in function of its observed outputs Y_1^n. Moreover, each of the two users $k = 1, 2$ attempts to decode its intended message M_k by producing a guess $\widehat{M}_k$. User 1 produces its guess based on its observed outputs Y_1^n only while User 2 produces its own based on both its observed outputs Y_2^n and the cooperation message L_c.

Accordingly, we have the following definitions:

Definition 1. An $(n, \nu_1, \nu_2, 2^{nC_{12}}, \varepsilon)$ code for a cooperative BC consist of a source encoder

$$f : \mathcal{N}_1 \times \mathcal{N}_2 \to \mathcal{X}^n, \tag{1}$$

a cooperation encoder

$$f_c : \mathcal{Y}_1^n \to \mathcal{N}_c, \tag{2}$$

two decoders

$$\varphi_1 : \mathcal{Y}_1^n \to \mathcal{N}_1, \tag{3.1}$$

$$\varphi_2 : \mathcal{Y}_2^n \times \mathcal{N}_c \to \mathcal{N}_2, \tag{3.2}$$

and a threshold on the probability of error

$$\frac{1}{\nu_1 \nu_2} \sum_{(j_1,j_2) \in \mathcal{N}_1 \times \mathcal{N}_2} P^{\otimes n}_{Y_1 Y_2|X}\Big(D^c_{j_1 j_2} \,|f(j_1, j_2)\Big) \le \varepsilon, \tag{4}$$

where $D_{j_1 j_2}$ is the decision region of the messages (j_1, j_2) and $D^c_{j_1 j_2}$ its complement.

Definition 2. A pair of rates (R_1, R_2) is called achievable with cooperation capacity C_{12} if for any given $\delta > 0$, $\varepsilon > 0$ and sufficiently large blocklength n, there exists an $(n, 2^{n(R_1-\delta)}, 2^{n(R_2-\delta)}, 2^{nC_{12}}, \varepsilon)$ code.

The capacity region, denoted $\mathcal{C}(C_{12})$, is the closure of the set of all achievable pairs (R_1, R_2) for a given cooperation capacity $C_{12} \ge 0$.

For positive cooperation capacity $C_{12} > 0$, the capacity region $\mathcal{C}(C_{12})$ depends on the joint conditional law $P_{Y_1,Y_2|X}$, while without cooperation it depends only on the marginal transition probabilities $P_{Y_1|X}$ and $P_{Y_2|X}$.

A prominent and simple coding strategy is to use superposition coding (SPC) at the transmitter and decode-and-forward (D&F) at User 1. In this context, D&F relies on binning: the set $\mathcal{N}_2$ is divided into $2^{n\mathcal{N}_c}$ bins, each containing $2^{n[R_2-C_{12}]_+}$ elements. User 1 performs joint decoding of M_1 and M_2 then chooses the cooperation message as the bin number of its estimated M_2. As described in [7, Appx. B], this strategy achieves all rate pairs in $\mathcal{R}_{in}(C_{12})$, a region defined as the set of all pairs (R_1, R_2) satisfying

$$R_1 \le \mathrm{I}(X; Y_1|U), \tag{5.1}$$

$$R_2 \le \mathrm{I}(U; Y_2) + C_{12}, \tag{5.2}$$

$$R_1 + R_2 \le \mathrm{I}(X; Y_1), \tag{5.3}$$

for some joint distribution $P_{UX} P_{Y_1 Y_2\,|X}$.

The region $\mathcal{R}_{in}(C_{12})$ is known to coincide with the capacity region only in the special cases of PD CBCs [5] or less noisy BCs without cooperation. It also achieves capacity for arbitrary ABCs [8], [9].

A valid outer bound on capacity region for MC BCs is $\mathcal{R}_{out}(C_{12})$ [7, Eq. 69], defined as the pairs (R_1, R_2) satisfying

$$R_1 \le \mathrm{I}(X; Y_1|U), \tag{6.1}$$

$$R_2 \le \mathrm{I}(U; Y_2) + C_{12}, \tag{6.2}$$

for some joint distribution $P_{UX} P_{Y_1 Y_2\,|X}$.

Throughout the following we assume that the BC $P_{Y_1 Y_2|X}$ is MC, which means

$$I(X; Y_2) \le I(X; Y_1), \forall P_X, \tag{7}$$

and can thus write [7]:

$$\mathcal{R}_{in}(C_{12}) \subseteq \mathcal{C}(C_{12}) \subseteq \mathcal{R}_{out}(C_{12}). \tag{8}$$

Observe that both $\mathcal{R}_{in}(C_{12})$ and $\mathcal{R}_{out}(C_{12})$ depend only on $P_{Y_1|X}$ and $P_{Y_2|X}$ while $\mathcal{C}(C_{12})$ depends on $P_{Y_1,Y_2|X}$.

Note that the bounds (5) and (6) differ only in the additional sum-rate constraint (5.3). This fact was exploited in [7] to show that if for some range of rates (C_{12}, R_1, R_2) (5.1) and (5.2) imply (5.3), then inner and outer bounds coincide, yielding a capacity characterization for that range. The main contribution in [7] is to find statistical conditions (called strong LN conditions) under which the sum-rate bound (5.3) is implied by the individual rate bounds. In the present work, we exploit parametric characterizations of inner and outer bounds to identify such range of rates (C_{12}, R_1, R_2).

III. Results

Let C_1 and C_2 denote the marginal capacities of the channels $P_{Y_1|X}$ and $P_{Y_2|X}$. For any $R_1 \le C_1$, let $R_2^*(R_1)$ denote the largest rate R_2 so that (R_1, R_2) are achievable:

$$R_2^*(R_1) = \sup\{R_2 : (R_1, R_2) \in \mathcal{C}(C_{12})\}. \tag{9}$$

We start by stating a general result, which we then specialize to Gaussian BCs and to BCs that consist of a BEC to the stronger User 1 and a BSC to the weaker User 2.

A. General Result

Proposition 1. Consider a channel $P_{Y_1 Y_2|X}$ satisfying the following condition: regions $\mathcal{R}_{in}(C_{12})$ and $\mathcal{R}_{out}(C_{12})$ can both be expressed in parametric form of a common parameter α, as:

$$\mathcal{R}_{in}(C_{12}) = \bigcup_{\alpha \in [0,b]} \mathcal{R}^{\alpha}_{in}(C_{12}), \tag{10}$$

and

$$\mathcal{R}_{out}(C_{12}) = \bigcup_{\alpha \in [0,b]} \mathcal{R}^{\alpha}_{out}(C_{12}), \tag{11}$$

where

$$\begin{aligned}\mathcal{R}_{in}^{\alpha}(C_{12}) = \{(R_1,R_2) : R_1 &\le f_1(\alpha)\\ R_2 &\le f_2(\alpha)\\ R_1 + R_2 &\le C_1 \quad \},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{R}_{out}^{\alpha}(C_{12}) = \{(R_1,R_2) : R_1 &\le f_1(\alpha)\\ R_2 &\le f_2(\alpha)\},\end{aligned} \tag{13}$$

where

- f_1 is a continuous and strictly *increasing* function with $f_1(0) = 0$ and $f_1(b) = C_1$;
- f_2 is a continuous and strictly *decreasing* function with $f_2(0) = C_2 + C_{12}$ and $f_2(b) = C_{12}$;
- the sum $f_1 + f_2$ is strictly increasing.

Assume further that

$$C_{12} \le C_1 - C_2, \tag{14}$$

and define $\alpha_{th} \le b$ as the unique solution in $[0, b]$ to

$$f_1(\alpha_{th}) + f_2(\alpha_{th}) = C_1 \tag{15}$$

and

$$R_{1,th} := f_1(\alpha_{th}). \tag{16}$$

Notice that (15) has the desired unique solution because the left-hand side of (15) is continuous and strictly increasing, for $\alpha_{th} = 0$ it evaluates to $C_2 + C_{12}$ which by Assumption (14) is smaller than C_1, and for $\alpha_{th} = b$ it evaluates $C_1 + C_{12}$ which is larger than C_1.

Then, all pairs (R_1, R_2) in the capacity region $\mathcal{C}(C_{12})$ with

$$R_1 \le R_{1,th} \tag{17}$$

are achieved using SPC and D&F at User 1. Thus, the inner bound $\mathcal{R}_{in}(C_{12})$ is tight in this regime.

In particular, for $R_1 \le R_{1,th}$:

$$R_2^*(R_1) = f_2(f_1^{-1}(R_1)), \tag{18}$$

and the maximum sum-rate is bounded as:

$$R_1 + R_2^*(R_1) \le C_1, \tag{19}$$

where the inequality is strict unless $R_1 = R_{1,th}$.

Remark 1. The thresholds α_{th} and $R_{1,th}$ are continuous and decreasing functions of C_{12} ranging from b and C_1, respectively, when $C_{12} = 0$, to 0 when $C_{12} = C_1 - C_2$.

Above proposition and remark are proved in Section IV.A.

In the following, we specialize above proposition to the Gaussian BC where User 1 has larger capacity than User 2, as well as to BCs consisting of a BEC to User 1 and a BSC to User 2 for all parameter ranges so that User 1 is MC than User 2. Notice that Proposition 1 provides also similar results for MC BCs consisting of two BSCs or two BECs. Details for these scenarios are omitted due to lack of space.

B. The Gaussian BC

Consider the Gaussian BC

$$Y_k = \sqrt{s_k} X + Z_k, \;\; k = 1, 2, \tag{20}$$

where Z_1 and Z_2 are jointly Gaussian of unit variances and s_1 and s_2 are given constants $s_1 > s_2 > 0$.

The capacity of the marginal channel $P_{Y_k|X}$ is $C_k = \Gamma(s_k)$, for $k = 1, 2$, where we define the Gaussian capacity function $\Gamma(x) = \log(1 + x)/2$. Note that the function Γ is invertible.

Define further, for $C_{12} \le C_1 - C_2$,

$$\alpha_{th} := \left(\frac{s_1 - s_2}{\Gamma^{-1}(C_1 - C_2 - C_{12})} - s_2 \right)^{-1} \tag{21}$$

and

$$R_{1,th} := \Gamma(\alpha_{th} s_1). \tag{22}$$

Theorem 1. Consider above Gaussian CBC and assume that

$$C_{12} \le C_1 - C_2. \tag{23}$$

All rate-pairs (R_1, R_2) in the capacity region $\mathcal{C}(C_{12})$ with

$$R_1 \le R_{1,th} \tag{24}$$

are achieved using SPC and D&F at User 1.

Furthermore, for $R_1 \le R_{1,th}$:

$$R_2^*(R_1) = C_2 + C_{12} - \Gamma\left(\Gamma^{-1}(R_1) \frac{s_2}{s_1} \right) \tag{25}$$

and the sum-rate is bounded by C_1:

$$R_1 + R_2^*(R_1) \le C_1, \tag{26}$$

where the inequality is strict unless $R_1 = R_{1,th}$.

Above theorem follows from Proposition 1. Details are provided in Section IV.B.

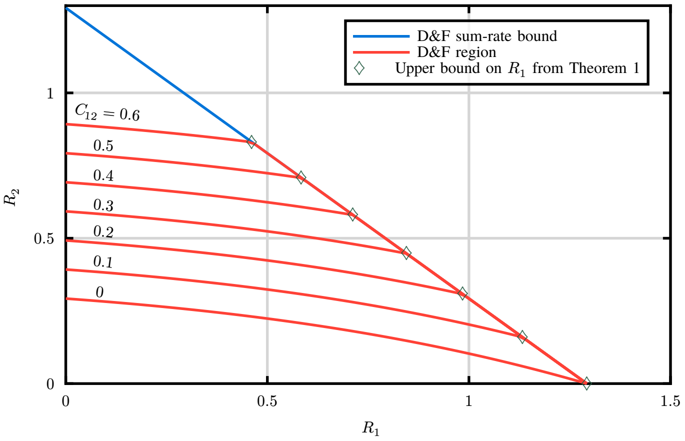

Fig. 2 depicts the region $\mathcal{R}_{in}(C_{12})$ for this Gaussian example when $s_1 = 5$, $s_2 = 0.5$, and different values of the cooperation capacity C_{12}. For positive values of $C_{12} > 0$, our theorem shows that it is optimal for all boundary points with sum-rate $R_1 + R_2 < C_1$ and the left-most point with $R_1 + R_2 = C_1$. This limiting boundary point is indicated with a diamond symbol in the figure. For all boundary points of $\mathcal{R}_{in}(C_{12})$ to the right, the sum-rate equals C_1. We conjecture that these points lie in the interior of the capacity region, and not on its boundary, and thus that SPC and D&F at User 1 are optimal if, and only if, $R_1 + R_2 \le C_1$.

C. *The BC with a BEC to User 1 and a BSC to User 2*

We next consider a BEC(τ_1) to User 1 and a BSC(p_2) to User 2, where we assume $p_2 \in [0, 1/2)$ and moreover

$$0 \le \tau \le \mathrm{H_b}(p_2). \tag{27}$$

These conditions ensure that the BEC is MC than the BSC [1, Ex. 5.4].

For $C_{12} \le C_1 - C_2$, define now q_{th} as the unique solution in $[0, 1/2]$ to the equation

$$\mathrm{H_b}(p_2 \star q_{th}) - \mathrm{H_b}(q_{th})(1-\tau_1) = C_{12} + \tau_1, \tag{28}$$

where $\mathrm{H_b}$ denotes the binary entropy function. Define further

$$R_{1,th,2} := \mathrm{H_b}(q_{th})(1-\tau_1). \tag{29}$$

Theorem 2. Consider above CBC formed by a BEC(τ_1) to User 1 and a BSC(p_2) to User 2 and assume that

$$C_{12} \le C_1 - C_2. \tag{30}$$

All rate-pairs (R_1, R_2) in the capacity region $\mathcal{C}(C_{12})$ with

$$R_1 \le R_{1,th,2} \tag{31}$$

are achieved using SPC and D&F at User 1.

Furthermore, for $R_1 \le R_{1,th,2}$:

$$R_2^*(R_1) = 1 - \mathrm{H_b}\left(p_2 \star \mathrm{H_b^{-1}}\left(\frac{R_1}{1-\tau_1}\right)\right) + C_{12}, \tag{32}$$

where $\mathrm{H_b^{-1}}$ denotes the inverse of the binary entropy function on the domain $[0, 1/2]$.

For all $R_1 \le R_{1,th,3}$, the sum-rate is bounded by C_1:

$$R_2^*(R_1) + R_1 \le C_1, \tag{33}$$

where the inequality is strict except for $R_1 = R_{1,th,2}$.

The proof is again obtained from Proposition 1 and details are given in Section IV.C.

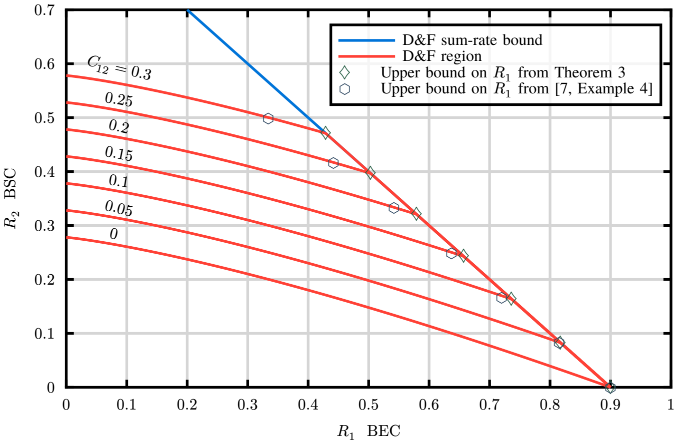

Fig. 3 shows the region $\mathcal{R}_{in}(C_{12})$ for a BC consisting of a BEC(0.1) to User 1 and a BSC(0.2) to User 2, for different values of the cooperation capacity C_{12}. Our theorem shows that $\mathcal{R}_{in}(C_{12})$ coincides with the capacity region for all boundary points with sum-rate $R_1 + R_2 < C_1$, as well as the left-most boundary point where $R_1 + R_2 = C_1$. This limiting boundary point is indicated with a diamond symbol in the figure. The hexagon symbols indicate the limiting points were the previous work [7] showed optimality of $\mathcal{R}_{in}(C_{12})$ (and thus of SPC and D&F).

From the figure we see that the current work establishes the optimality of SPC and D&F on a larger part of the capacity region compared to the results in [7, Thm. 1]. Note that we also enlarge the class of channels and cooperation rates C_{12} for which D&F is optimal.

IV. PROOFS

A. *Proof of Proposition 1*

Define for each $R_1 \le C_1$:

$$\bar{R}_2(R_1) := \sup\{R_2 : (R_1, R_2) \in \mathcal{R}_{out}(C_{12})\}. \tag{34}$$

We start by proving that

$$\mathcal{R}_{in}^{\alpha}(C_{12}) = \mathcal{R}_{out}^{\alpha}(C_{12}), \forall \alpha \le \alpha_{th}. \tag{35}$$

To this end, notice that $\mathcal{R}_{out}^{\alpha}(C_{12})$ is a rectangular region, and $\mathcal{R}_{in}^{\alpha}(C_{12})$ coincides with this rectangular region whenever its sum-rate constraint is not active:

$$f_1(\alpha) + f_2(\alpha) \le C_1. \tag{36}$$

Since the sum $f_1 + f_2$ is a continuous and strictly increasing function, this inequality is satisfied for all $\alpha \le \alpha_{th}$, establishing the desired Equality (35).

Moreover, since $f_1 + f_2$ is strictly increasing, it can be deduced that for all $\alpha \le \alpha_{th}$ it holds that $f_1(\alpha) + f_2(\alpha) \le C_1$, where the inequality is strict unless $\alpha = \alpha_{th}$. This implies that the sum-rate of the dominant corner point (and thus of all points) of $\mathcal{R}_{out}^{\alpha}(C_{12})$ is bounded by C_1, where equality only holds when $\alpha = \alpha_{th}$.

We next argue that each boundary point $(R_1, \bar{R}_2(R_1))$ of $\mathcal{R}_{out}(C_{12})$ with $R_1 \le R_{1,th}$ is the dominant corner point of the rectangle $\mathcal{R}_{out}^{\alpha(R_1)}(C_{12})$ for

$$\alpha(R_1) := f_1^{-1}(R_1). \tag{37}$$

Since $\alpha(R_{1,th}) = \alpha_{th}$, and f_1^{-1} is continuous and strictly increasing, we can then deduce by (35) that $(R_1, \bar{R}_2(R_1))$ also lies in $\mathcal{R}_{in}^{\alpha(R_1)}(C_{12})$ and is thus achievable using SPC and D&F. To see that $(R_1, \bar{R}_2(R_1))$, with $R_1 \le R_{1,th}$, is the dominant corner point of the rectangle $\mathcal{R}_{out}^{\alpha(R_1)}(C_{12})$, it suffices to notice that the largest R_1 in $\mathcal{R}_{out}^{\alpha}(C_{12})$ is continuous and strictly increasing in α and the largest R_2 continuous and strictly decreasing in α. This establishes the first part of the proposition.

We continue by noticing that from the considerations around (36), we can conclude that for all $R_1 \le R_{1,th}$ we have

$$R_1 + \bar{R}_2(R_1) \le C_1, \tag{38}$$

because $\alpha(R_1) \le \alpha_{th}$. And moreover, the inequality is strict unless $\alpha(R_1) = \alpha_{th}$ and thus $R_1 = R_{1,th}$.

To establish the last two desired statements (18) and (19) of the proposition, it suffices to notice that $(R_1, \bar{R}_2(R_1))$ coincides with $(R_1, R_2^*(R_1))$ because $\mathcal{R}_{out}^{\alpha}(C_{12})$ lies on the boundary of an outer bound. We therefore, have by the definition of $\mathcal{R}_{out}^{\alpha}(C_{12})$,

$$R_2^*(R_1) = f_2(f_1^{-1}(R_1)). \tag{39}$$

and by (38),

$$R_1 + R_2^*(R_1) \le C_1. \tag{40}$$

B. Proof of Theorem 1

We verify that the conditions in Proposition 1 are satisfied.

Using standard arguments and the Entropy Power Inequality, similarly to the non-cooperative case, it can be shown that jointly Gaussian P_{UX} exhaust the regions $\mathcal{R}_{in}(C_{12})$ and $\mathcal{R}_{out}(C_{12})$ [1, Chap. 5.5]. Therefore, the following parametric forms hold:

$$\mathcal{R}_{in}(C_{12}) = \bigcup_{\alpha \in [0,1]} \mathcal{R}^{\alpha}_{in}(C_{12}), \tag{41}$$

and

$$\mathcal{R}_{out}(C_{12}) = \bigcup_{\alpha \in [0,1]} \mathcal{R}^{\alpha}_{out}(C_{12}), \tag{42}$$

for

$$\begin{aligned} \mathcal{R}^{\alpha}_{in}(C_{12}) := \{(R_1, R_2) : R_1 &\leq \Gamma(\alpha s_1) \\ R_2 &\leq C_2 + C_{12} - \Gamma(\alpha s_2) \\ R_1 + R_2 &\leq C_1 \quad \}, \end{aligned} \tag{43}$$

and

$$\begin{aligned} \mathcal{R}^{\alpha}_{out}(C_{12}) := \{(R_1, R_2) : R_1 &\leq \Gamma(\alpha s_1) \\ R_2 &\leq C_2 + C_{12} - \Gamma(\alpha s_2)\}. \end{aligned} \tag{44}$$

Notice that the function $f_1(\alpha) = \Gamma(\alpha s_1)$ is continuous, strictly increasing, and satisfies $f_1(0) = 0$ and $f_1(1) = C_1$. Similarly, the function $f_2(\alpha) = C_2 + C_{12} - \Gamma(\alpha s_2)$ is continuous, strictly decreasing, and satisfies $f_2(0) = C_2 + C_{12}$ and $f_2(1) = C_{12}$. Moreover, the sum

$$f_1(\alpha) + f_2(\alpha) = \frac{1}{2}\log\left(\frac{1 + \alpha s_1}{1 + \alpha s_2}\right) \tag{45.1}$$

$$= C_2 + C_{12} + \frac{1}{2}\log\left(1 + \frac{s_1 - s_1}{\frac{1}{\alpha} + s_2}\right) \tag{45.2}$$

is strictly increasing in α.

We thus conclude that above parametric form satisfies all the conditions in Proposition 1.

It remains to verify that the proposed values for α_{th} and $R_{1,th}$ as well as for $R_2^*(R_1)$ are as in Proposition 1. By Proposition 1, α_{th} is the unique solution to

$$\Gamma(\alpha_{th} s_1) + C_2 + C_{12} - \Gamma(\alpha_{th} s_2) = C_1, \tag{46}$$

which is equivalent to

$$C_1 - C_2 - C_{12} = \frac{1}{2}\log\left(1 + \frac{s_1 - s_2}{\frac{1}{\alpha_{th}} + s_2}\right) \tag{47}$$

and can be rewritten as

$$\alpha_{th} = \left(\frac{s_1 - s_2}{\Gamma^{-1}(C_1 - C_2 - C_{12})} - s_2\right)^{-1}, \tag{48}$$

which corresponds to the definition in Theorem 1. Moreover, $R_{1,th} = f_1(\alpha_{th}) = \Gamma(\alpha_{th} s_1)$ as indicated in the theorem.

Finally, by Proposition 1,

$$R_2^*(R_1) = f_2(f_1^{-1}(R_1)) \tag{49.1}$$

$$= C_2 + C_{12} - \Gamma\left(\Gamma^{-1}(R_1)\frac{s_2}{s_1}\right). \tag{49.2}$$

This concludes the proof of Theorem 1.

C. Proof of Theorem 2

We verify that the conditions in Proposition 1 are satisfied.

Using standard arguments, involving Mrs. Gerber's Lemma [11], it can be shown (for details, see [12, Appx. A]) that

$$\mathcal{R}_{in}(C_{12}) = \bigcup_{q \in [0,1/2]} \mathcal{R}^{q}_{in}(C_{12}), \tag{50}$$

and

$$\mathcal{R}_{out}(C_{12}) = \bigcup_{q \in [0,1/2]} \mathcal{R}^{q}_{out}(C_{12}). \tag{51}$$

where for given $q \in [0, 1/2]$:

$$\begin{aligned} \mathcal{R}^{q}_{in}(C_{12}) := \{(R_1, R_2) : R_1 &\leq \mathrm{H_b}(q)(1 - \tau_1) \\ R_2 &\leq 1 - \mathrm{H_b}(p_2 \star q) + C_{12} \\ R_1 + R_2 &\leq 1 - \tau_1 \quad \}, \end{aligned} \tag{52}$$

and

$$\begin{aligned} \mathcal{R}^{q}_{out}(C_{12}) := \{(R_1, R_2) : R_1 &\leq \mathrm{H_b}(q)(1 - \tau_1) \\ R_2 &\leq 1 - \mathrm{H_b}(p_2 \star q) + C_{12}\}. \end{aligned} \tag{53}$$

Notice that the function

$$f_1(q) = \mathrm{H_b}(q)(1 - \tau_1) \tag{54}$$

is continuous and strictly increasing in q, with $f_1(0) = 0$ and $f_1(1/2) = C_1$. Similarly, the function

$$f_2(q) = 1 - \mathrm{H_b}(p_2 \star q) + C_{12} \tag{55}$$

is continuous and strictly decreasing in q, with $f_2(0) = C_2 + C_{12}$ and $f_2(1/2) = C_{12}$. Moreover, the sum

$$\begin{aligned} &f_1(q) + f_2(q) \\ &\quad = \mathrm{H_b}(q)(1 - \tau_1) + 1 - \mathrm{H_b}(p_2 \star q) + C_{12} \end{aligned} \tag{56}$$

is strictly increasing in q as can be seen by examining the derivative.

We thus conclude that above parametric form satisfies the conditions in Proposition 1. It remains to verify that the values for q_{th} and $R_{1,th,2}$, as well as for $R_2^*(R_1)$ proposed in the theorem are as in Proposition 1.

By Proposition 1, q_{th} is the unique solution to

$$\mathrm{H_b}(q)(1 - \tau_1) + 1 - \mathrm{H_b}(p_2 \star q) + C_{12} = C_1 \tag{57}$$

which is equivalent to the expression proposed in the theorem. Furthermore, $R_{1,th} = f_1(q_{th})$, also as proposed in the theorem. Finally, by Proposition 1:

$$R^*(R_1) = f_2(f_1^{-1}(R_1)) \tag{58.1}$$

$$= 1 - \mathrm{H_b}\left(p_2 \star \mathrm{H_b}^{-1}\left(\frac{R_1}{1 - \tau_1}\right)\right) + C_{12}. \tag{58.2}$$

This concludes the proof.

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