Coding and Limits for Wireless Distributed Computing

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Outline

- **Part I: Partially-Connected Networks**
  - General interference alignment (IA) coding scheme
  - A novel number-filling puzzle capturing the network structure
  - The score of the puzzle solution reflects the sum rate
- **Part II: Wireless Distributed Computing (DC)**
  - Focus on MapReduce-based wireless systems
  - Exploiting structural parallels to apply IA scheme and enhance communication-computation tradeoff
  - Information-theoretic converse bound proving near-optimality

Why Partially-Connected Networks?

- **Realistic Modeling:** Many practical systems (e.g., wireless, IoT, and fog networks) exhibit limited connectivity due to physical constraints, interference, or energy limits.

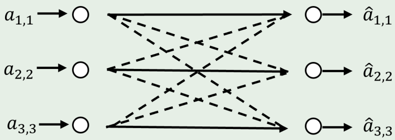
- **Broad Applications:** Distributed computing, D2D caching, and cooperative communication systems.

- **New Theoretical Questions:** Partial connectivity introduces new design and analysis challenges, e.g. topology-aware interference management, resource allocation.

Partially Connected Channel

- Channel with K Txs and K Rxs.
- $\mathbf{N} \in \{0, 1\}^{K \times K}$: connectivity matrix.
- $\mathbf{M} \in \{0, 1\}^{K \times K}$: message flow matrix.

Example (Interference channel with $K = 3$)

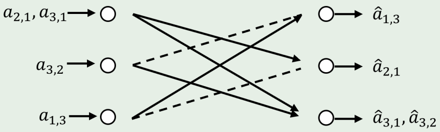


Corresponding matrices:

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Partially Connected Channel

Example (A partially connected channel with $K = 3$)



Corresponding matrices:

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \qquad .$$

- Rx p observes

$$Y_p(t) = \sum_{\{q:\ \mathbf{N}[p,q]=1\}} H_{p,q}(t) X_q(t) + Z_p(t), \quad p \in [K],\ t \in [T]$$

  - T is the total blocklength.
  - $\{H_{p,q}(t)\}$ are i.i.d. fading according to a bounded continuous distribution.
  - Assume *perfect CSI*

Partially Connected Channel

- Tx q transmits a message $a_{p,q}$ to Rx p for which $\mathbf{M}[p, q] = 1$.

$$\mathbf{X}_q \triangleq (X_q(1), \cdots, X_q(\mathrm{T})) = f_q^{(\mathrm{T})}(\{a_{p,q} \colon \mathbf{M}[p,q] = 1\})$$

- Each Tx produces inputs with a power constraint P.
- Message $a_{p,q}$ is uniformly distributed over $\left[2^{\mathrm{TR}_{p,q}}\right]$.

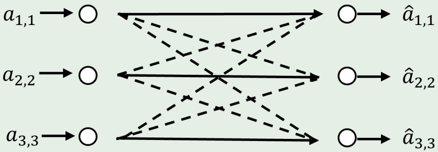
- *Sum Degrees of Freedom (SDoF)*:

$$\mathrm{SDoF} \triangleq \overline{\lim_{\mathrm{P}\to\infty}} \sup_{\mathbf{R}\in\mathcal{C}(\mathrm{P})} \sum_{\mathbf{M}[p,q]=1} \frac{\mathrm{R}_{p,q}}{\log \mathrm{P}}.$$

Capacity region $\mathcal{C}(\mathrm{P})$ is set of tuple $(\mathrm{R}_{p,q})$ s.t. $p(error) \to 0$ for given power P.

Interference Alignment (IA) scheme for Interference Channels

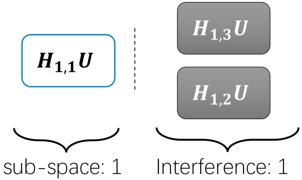
Example (An interference channel with K = 3)



- Encode each message $a_{q,q}$ to η^{Γ}-length codeword $\mathbf{b}_{q,q}$, and pre-multiply $\mathbf{U}$:

$$\mathbf{X}_q = \mathbf{U}\mathbf{b}_{q,q}$$

- By designing **U**, Rx1 subspace (Rx2, Rx3 are similar):



- We choose $\mathcal{H}$ as the set of all channel coefficients of interference links: $\mathbf{H}_{1,2}, \mathbf{H}_{1,3}, \mathbf{H}_{2,1}, \mathbf{H}_{2,3}, \mathbf{H}_{3,1}, \mathbf{H}_{3,2}$.
- The matrix $\mathbf{U}$ is

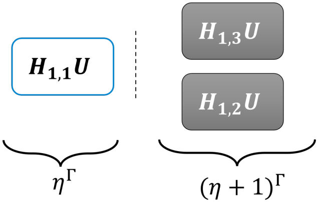
$$\mathbf{U} = \left[\mathbf{1}, \mathbf{H}_{1,2}\mathbf{1}, \mathbf{H}_{1,2}^2\mathbf{1}, \ldots, \mathbf{H}_{1,2}^{\eta}\mathbf{1}, \mathbf{H}_{1,2}\mathbf{H}_{1,3}\mathbf{1}, \ldots \mathbf{H}_{1,2}^{\eta}\mathbf{H}_{1,3}^{\eta}\mathbf{1}, \ldots\right]$$

$$= \left[\prod_{\mathbf{H}_{p,q}\in\mathcal{H}} \mathbf{H}_{p,q}^{\alpha_{p,q}} \cdot \mathbf{1} : \ \forall \boldsymbol{\alpha} \in [\eta]^{\Gamma}\right], \quad \text{with } \Gamma = |\mathcal{H}|.$$

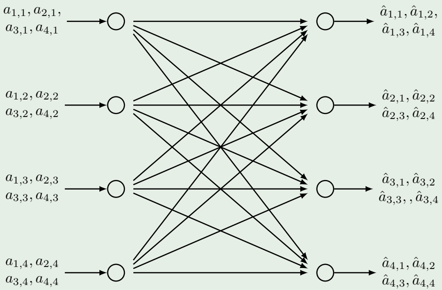
- For all $\mathbf{H}_{p,q} \in \mathcal{H}$,

$$\text{span}(\mathbf{HU}) \subseteq \text{span}(\mathbf{W}), \quad \mathbf{W} = \left[\prod_{\mathbf{H}_{p,q}\in\mathcal{H}} \mathbf{H}_{p,q}^{\alpha_{p,q}} \cdot \mathbf{1} : \ \forall \boldsymbol{\alpha} \in [\eta+1]^{\Gamma}\right].$$

- For all $\mathbf{H}_{p,p} \notin \mathcal{H}$, e.g. $\mathbf{H}_{1,1}$, $\mathbf{HU}$ has full column rank, span($\mathbf{HU}$) and span($\mathbf{W}$) are **linearly independent**.



Example (X-channel with $K = 4$)



Corresponding matrices:

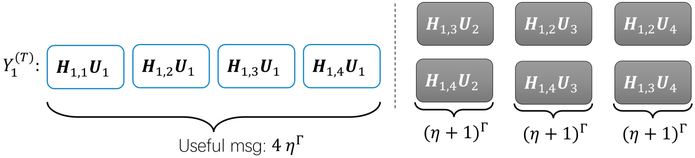
$$\mathbf{M} = \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

- Encode $a_{p,q}$ to η^Γ-length codeword $\mathbf{b}_{p,q}$.

- $\mathbf{U}_p = \left[\prod_{\mathbf{H}\in\mathcal{H}_p} \mathbf{H}^{\alpha_\mathbf{H}} \cdot \mathbf{\Xi}_p : \ \ \forall \boldsymbol{\alpha} \in [\eta]^\Gamma\right]$ multiply $\{\mathbf{b}_{p,1}, ..., \mathbf{b}_{p,4}\}$.

$$
\begin{aligned}
X_1^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,1} + \mathbf{U}_2\mathbf{b}_{2,1} + \mathbf{U}_3\mathbf{b}_{3,1} + \mathbf{U}_4\mathbf{b}_{4,1},\\
X_2^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,2} + \mathbf{U}_2\mathbf{b}_{2,2} + \mathbf{U}_3\mathbf{b}_{3,2} + \mathbf{U}_4\mathbf{b}_{4,2},\\
X_3^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,3} + \mathbf{U}_2\mathbf{b}_{2,3} + \mathbf{U}_3\mathbf{b}_{3,3} + \mathbf{U}_4\mathbf{b}_{4,3},\\
X_4^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,4} + \mathbf{U}_2\mathbf{b}_{2,4} + \mathbf{U}_3\mathbf{b}_{3,4} + \mathbf{U}_4\mathbf{b}_{4,4}.
\end{aligned}
$$

- By choose matrices $\mathbf{U}_p$ such that [1]: (Rx 1, 2, 3, 4 are similar)



[1] V. R. Cadambe and S. A. Jafar, "Interference Alignment and the Degrees of Freedom of Wireless *X* Networks," IEEE Trans. Inform. Theory, Sep. 2009.

Matrix representation of IA schemes

- A matrix $\mathbf{G}$ is applied to describe the precoding matrix allocation.
  - $\mathbf{G}[p,q] = \mathbf{G}[p',q'] = g > 0$: $\mathbf{b}_{p,q}$ and $\mathbf{b}_{p',q'}$ are premultiplied by the same $\mathbf{U}_g$.
  - $\mathbf{G}[p,q] = 0$: $\mathbf{b}_{p,q}$ is not transmitted.

$$
\begin{aligned}
X_1^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,1} + \mathbf{U}_2\mathbf{b}_{2,1} + \mathbf{U}_3\mathbf{b}_{3,1} + \mathbf{U}_4\mathbf{b}_{4,1},\\
X_2^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,2} + \mathbf{U}_2\mathbf{b}_{2,2} + \mathbf{U}_3\mathbf{b}_{3,2} + \mathbf{U}_4\mathbf{b}_{4,2},\\
X_3^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,3} + \mathbf{U}_2\mathbf{b}_{2,3} + \mathbf{U}_3\mathbf{b}_{3,3} + \mathbf{U}_4\mathbf{b}_{4,3},\\
X_4^{(T)} &= \mathbf{U}_1\mathbf{b}_{1,4} + \mathbf{U}_2\mathbf{b}_{2,4} + \mathbf{U}_3\mathbf{b}_{3,4} + \mathbf{U}_4\mathbf{b}_{4,4}.
\end{aligned}
$$

$$\Downarrow$$

$$
\mathbf{G} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}
$$

Calculate SDoF with Matrix representation

- $\|\mathbf{G}[:,p]\|_0$: the number of non-zero entries in the p-th column (codewords for Rx p) of $\mathbf{G}$.
- $g^{(p)}$: the number of different non-zeros integers in the interfering codeword submatrix, which is
$$\mathbf{G}^{(p)} \triangleq \mathbf{G}[p' \neq p, \{q\colon \mathbf{N}[p,q] = 1\}].$$
- DoF of Rx p is given by:
$$\text{DoF}_p = \frac{\|\mathbf{G}[p,:]\|_0}{\|\mathbf{G}[p,:]\|_0 + g^{(p)}},$$

For $p = 1$ ($p \in \{2,3,4\}$ are similar),

$$\mathbf{G}[:,1] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \|\mathbf{G}[:,1]\|_0 = 4, \quad \mathbf{G}^{(1)} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow g^{(1)} = 3.$$

SDoF = $\frac{16}{4+3} = 16/7$.

Main Result: Correspondence between **G**s and IA schemes

G should satisfy the following two requirements:

- **G** inherits zeros from **M**. I.e., $\mathbf{M}[p,q]=0 \Rightarrow \mathbf{G}[p,q]=0$.
- All non-zero entries in a column are different. I.e., $\mathbf{G}[p,q]=\mathbf{G}[p',q] \Rightarrow \mathbf{G}[p,q]=0$.

Theorem

*For an $(\mathbf{M},\mathbf{N})$-channel and a matrix **G** satisfying the above two points, Sum-DoF of the network is lower bounded by:*

$$\text{Sum-DoF} \geq \frac{\|\mathbf{G}\|_0}{\max_{p\in[\mathsf{K}]}\left\{\|\mathbf{G}[p,:]\|_0 + g^{(p)}\right\}},$$

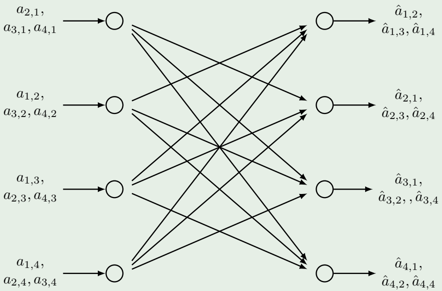
where

- $\|\mathbf{G}[p,:]\|_0$*: the number of non-zero entries in the p-th row of* **G**.
- $g^{(p)}$*: the number of different non-zeros integers in the submatrix*

$$\mathbf{G}^{(p)} \triangleq \mathbf{G}[p' \neq p, \{q\colon \mathbf{N}[p,q]=1\}].$$

Partially Connected Channel

Example (Partially-connected X-channel)



Corresponding matrices:

$$\mathbf{M} = \mathbf{N} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Naive Scheme for Partially-Connected Channel with $K = 4$

Example (Classic IA)

$$\mathbf{M} = \mathbf{N} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 \\ 1 & 2 & 3 & 0 \end{bmatrix},$$

For $p = 1$ ($p \in \{2, 3, 4\}$ are similar),

$$\|\mathbf{G}[:, 1]\|_0 = 3, \quad \mathbf{G}^{(1)} = \begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix} \Rightarrow g^{(1)} = 3.$$

Sum-DoF = $\frac{12}{3+3} = 2$.

- To ensure decodablility, all non-zero entries in a row should be **different**.

Example (IA with new allocation scheme)

- Encode all message except $(p, q) = (3, 4)$.
- Only apply 3 precoding matrices.

$$\mathbf{M} = \mathbf{N} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

For $p = 1$ ($p \in \{2, 3\}$ are similar),

$$\|\mathbf{G}[:, 1]\|_0 = 3, \quad \mathbf{G}^{(1)} = \begin{bmatrix} 0 & 3 & 2 \\ 2 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix} \Rightarrow g^{(1)} = 2.$$

For $p = 4$,

$$\|\mathbf{G}[:, 4]\|_0 = 2, \quad \mathbf{G}^{(4)} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \Rightarrow g^{(4)} = 3.$$

Sum-DoF = $\frac{11}{2+3} = 11/5$.

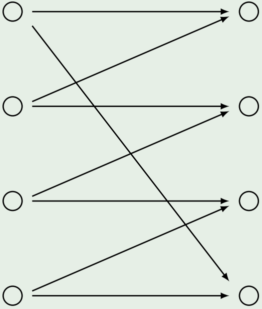
General Results for Symmetric Regular Channels

- Consider symmetric channels with K user, where each Tx is connected to $m = n \in [K]$ Rx(s), and each Tx sends an independent message to each connected Rx:

$$\mathbf{N}[p,q] = \begin{cases} 1 & \text{if } (p-q) \mod K < n \\ 0 & \text{otherwise} \end{cases}, \quad \mathbf{M}[p,q] = \begin{cases} 1 & \text{if } (p-q) \mod K < m \\ 0 & \text{otherwise} \end{cases}.$$

Example (An example with $K = 4$, $m = n = 2$)

$$\mathbf{M} = \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$



Numerical Results for Symmetric Regular Channels

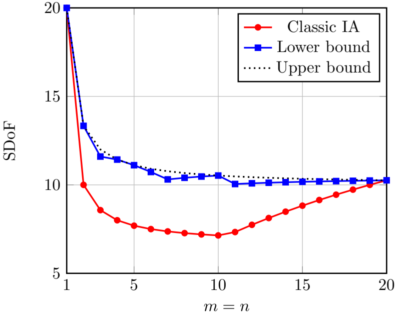
Theorem (Bounds for regular channels)

For the channel with matrices **M**, **N** $m < n$, Sum-DoF $= \frac{K}{2}$.
For $m = n$, *we have the following lower bound:*

$$\text{Sum-DoF}_{\text{Lb}} \geq \frac{Km - (K \bmod m) + \epsilon}{2m - 1} \quad \text{,where } \epsilon = \begin{cases} 1, & m = (K+1)/2 \text{ and } K \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Upper bound:

$$\text{Sum-DoF}_{\text{Lb}} = \frac{K \cdot m}{2m - 1}.$$



Summary and Outlook of Part I

- Introduced a novel number-filling puzzle as an intuitive design tool for IA scheme.
- Proposed a improved IA scheme for partially-connected networks.

Outlook:

- Extend the "puzzle" to networks with cooperation

Part II: Distributed Systems

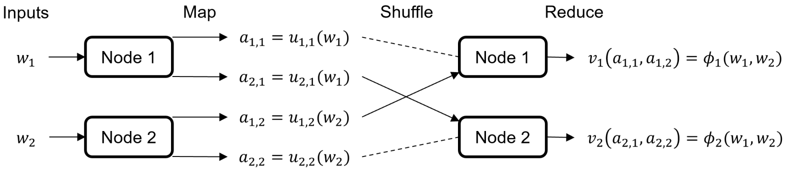
- Distributed system is a group of networked computers. These computers communicate and coordinate their actions to achieve a common goal.

- As components in distributed systems are physically separated and wireless smart devices have become increasingly dominant, studying wireless coding schemes for distributed systems is crucial.
  - Improved wireless coding schemes can significantly reduce the overall execution time of distributed computing systems.
  - Maintaining accuracy is critical for distributed estimation systems.
  - Efficient data synchronization is essential for distributed storage systems to ensure consistency and reliability.

- $\phi_k(w_1, \cdots w_N) = v_k(\underbrace{u_{k,1}(w_1)}_{a_{k,1}}, \cdots \underbrace{u_{k,N}(w_N)}_{a_{k,N}}), \quad k \in [Q]$
- **Map phase**: $\mathcal{M}_k \subseteq [N]$ is assigned to node k, it computes intermediate values (IVA) $a_{k,n}$.
- **Shuffle phase**: Exchange IVAs over wireless channel. Assume nodes are **full-duplex**.

$$Y_k(t) = \sum_{\ell \in [K] \setminus k} H_{k,\ell}(t) X_\ell(t) + Z_k(t), \quad t \in [T],$$

- **Reduce phase**: Apply reduce functions.



Wireless DC System Model: Performance Measures

The performance is measured by two factors:

- The *computation load* $\mathsf{r} \in [1, \mathsf{K}]$

$$\mathsf{r} \triangleq \sum_{p \in [\mathsf{K}]} \frac{|\mathcal{M}_p|}{\mathsf{N}},$$

- The *normalized delivery time (NDT)* $\Delta \in [0, 1]$

$$\Delta \triangleq \lim_{\mathsf{P} \to \infty} \lim_{\mathsf{A} \to \infty} \frac{\mathsf{T}}{\mathsf{Q} \cdot \mathsf{N} \cdot \mathsf{A} / \log \mathsf{P}}.$$

Objective: optimal computation-NDT tradeoff $\Delta^*(\mathsf{r})$

$$\Delta^*(\mathsf{r}) \triangleq \min_{\{f_p^{(\mathsf{T})}, g_{q,i}^{(\mathsf{T})}\}, \{\mathcal{M}_p\}} \Delta$$

Lemma of NDT and SDoF

Sum Degrees of Freedom (SDoF):

$$\mathrm{SDoF} \triangleq \varlimsup_{\mathrm{P}\to\infty} \sup_{\mathbf{R}\in\mathcal{C}(\mathrm{P})} \sum_{k\in[\tilde{\mathrm{K}}]} \sum_{p\in[\mathrm{K}]\setminus\mathcal{T}_k} \frac{\mathrm{R}_{p,k}(\mathrm{P})}{\log \mathrm{P}}.$$

Capacity region $\mathcal{C}(\mathrm{P})$ is set of tuple $(\mathrm{R}_{p,k}(\mathrm{P}))$ s.t. $p(error) \to 0$.

- Apply our IA scheme for the channel.

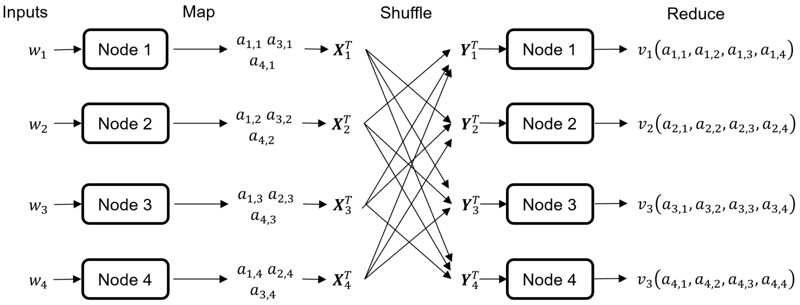
Lemma

Q *is a multiple of* K*. As* $N \to \infty$*,*

$$\Delta^*(\mathrm{r}) \le \mathrm{lowc}\left((\mathrm{K},0) \cup \left\{\left(\mathrm{r}, \frac{1-\mathrm{r}/\mathrm{K}}{\mathrm{SDoF}_{\mathrm{Lb}}}\right) : 1 \le \mathrm{r} < \mathrm{K} \text{ and } \mathrm{r}|\mathrm{K}\right\}\right),$$

where lowc$(\cdot)$ *denotes the lower-convex envelope.*

IA for Wireless DC System Model with r = 1



- **File assignment & Map phase**:Each node is assigned an output function and an input file.
- **Shuffle phase**: Channel is equivalent to a partially connected channel with

$$\mathbf{M} = \mathbf{N} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

- Assign a distinct input file to each group of r nodes. [2]

- Redesigned the precoding matrix allocation scheme.

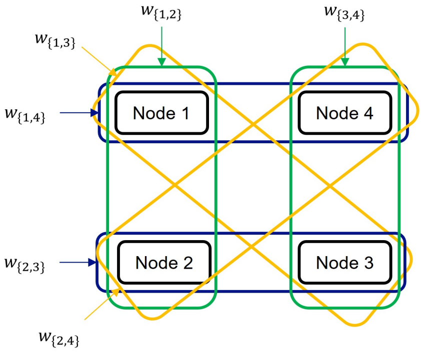


Figure: An example with K $= 4$, $r = 2$

[2] S. Li et al., "A Fundamental Tradeoff Between Computation and Communication in Distributed Computing," IEEE Trans. Inform. Theory, Jan. 2018.

Proposed Scheme for K = 4, r = 2: Encoding

- Send all messages $M^j_{k,\mathcal{T}}$ except all message from node 4 to node 1 i.e. $j = 1, k = 4$.
  - To Node 1 we send

$$M^1_{2,\{2,3\}},\ M^1_{3,\{2,3\}},\ M^1_{2,\{2,4\}},\ M^1_{3,\{3,4\}};$$

  - To Node 2 we send

$$M^2_{1,\{1,3\}},\ M^2_{3,\{1,3\}},\ M^2_{1,\{1,4\}},\ M^2_{4,\{1,4\}} M^2_{3,\{3,4\}}\ M^2_{4,\{3,4\}};$$

  - To Node 3 we send

$$M^3_{1,\{1,2\}},\ M^3_{2,\{1,2\}},\ M^3_{1,\{1,4\}},\ M^3_{4,\{1,4\}}, M^3_{2,\{2,4\}},\ M^3_{4,\{2,4\}};$$

  - To Node 4 we send

$$M^4_{1,\{1,2\}},\ M^4_{2,\{1,2\}},\ M^4_{1,\{1,3\}},\ M^4_{3,\{1,3\}}, M^4_{2,\{2,3\}},\ M^4_{3,\{2,3\}}.$$

Proposed Scheme for K = 4, r = 2: Encoding

- Encode the message $M^j_{k,\mathcal{T}}$ to η^Γ-length Gaussian codeword $\mathbf{b}^j_{k,\mathcal{T}}$.
- $\mathbf{b}^j_{k,\mathcal{T}}$ is precoded by
  - if $j = 1$, $\mathbf{U}_\mathcal{T}$
  - if $1 \in \mathcal{T}$, $\mathbf{U}_{\mathcal{T}\setminus\{1\}\cup\{j\}}$
  - if $1 \notin \mathcal{T}$ and $j \neq 1$, $\mathbf{U}_{\mathcal{T}\setminus\{k\}\cup\{j\}}$

$$\mathbf{X}_1 = \mathbf{U}_{\{2,3\}}\left(\mathbf{b}^2_{1,\{1,3\}} + \mathbf{b}^3_{1,\{1,2\}}\right) + \mathbf{U}_{\{2,4\}}\left(\mathbf{b}^2_{1,\{1,4\}} + \mathbf{b}^4_{1,\{1,2\}}\right) + \mathbf{U}_{\{3,4\}}\left(\mathbf{b}^3_{1,\{1,4\}} + \mathbf{b}^4_{1,\{1,3\}}\right),$$

$$\mathbf{X}_2 = \mathbf{U}_{\{2,3\}}\left(\mathbf{b}^1_{2,\{2,3\}} + \mathbf{b}^3_{2,\{1,2\}}\right) + \mathbf{U}_{\{2,4\}}\left(\mathbf{b}^1_{2,\{2,4\}} + \mathbf{b}^4_{2,\{1,2\}}\right) + \mathbf{U}_{\{3,4\}}\left(\mathbf{b}^3_{2,\{2,4\}} + \mathbf{b}^4_{2,\{2,3\}}\right),$$

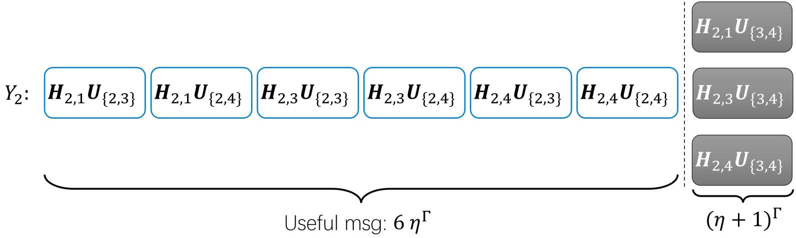
Proposed Scheme for K = 4, r = 2: Encoding

- Encode the message $a_{p,k}$ to η^{Γ}-length Gaussian codeword $\mathbf{b}_{p,k}$ except $(p,k) = (1,4)$.

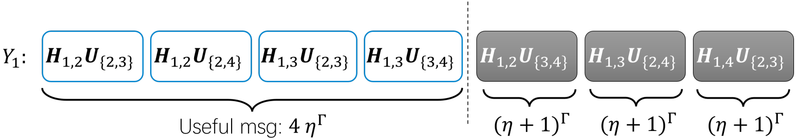
- $\mathbf{b}^{j}_{k,\mathcal{T}}$ is precoded by
  - if $j = 1$, $\mathbf{U}_{\mathcal{T}}$
  - if $1 \in \mathcal{T}$, $\mathbf{U}_{\mathcal{T}\setminus\{1\}\cup\{j\}}$
  - if $1 \notin \mathcal{T}$ and $j \neq 1$, $\mathbf{U}_{\mathcal{T}\setminus\{k\}\cup\{j\}}$

$$
\begin{aligned}
\mathbf{X}_3 = {} & \mathbf{U}_{\{2,3\}}\left(\mathbf{b}^{1}_{3,\{2,3\}} + \mathbf{b}^{2}_{2,\{1,3\}}\right) + \mathbf{U}_{\{2,4\}}\left(\mathbf{b}^{2}_{3,\{3,4\}} + \mathbf{b}^{4}_{3,\{2,3\}}\right) \\
& + \mathbf{U}_{\{3,4\}}\left(\mathbf{b}^{1}_{3,\{3,4\}} + \mathbf{b}^{4}_{3,\{1,3\}}\right), \\
\mathbf{X}_4 = {} & \mathbf{U}_{\{2,3\}}\left(\mathbf{b}^{2}_{4,\{3,4\}} + \mathbf{b}^{3}_{4,\{2,4\}}\right) + \mathbf{U}_{\{2,4\}}\mathbf{b}^{2}_{4,\{1,4\}} \\
& + \mathbf{U}_{\{3,4\}}\mathbf{b}^{3}_{4,\{1,4\}}
\end{aligned}
$$

- Choose matrices $\mathbf{U}_{\{2,3\}}, \mathbf{U}_{\{2,4\}}, \mathbf{U}_{\{3,4\}}$ such that [3]: (Rx 2,3,4 are similar)



- At Rx 1



- Choose $\mathrm{T} = 4\eta^{\Gamma} + 3(\eta+1)^{\Gamma}$

$$\mathrm{SDoF} = \lim_{\eta\to\infty} \frac{3\times 6\eta^{\Gamma} + 4\eta^{\Gamma}}{4\eta^{\Gamma} + 3(\eta+1)^{\Gamma}} \approx \frac{3\times 6}{7} + \frac{4}{7} = \frac{22}{7}$$

Upper Bound for Wireless DC System

Theorem

The computation-NDT tradeoff $\Delta^*(r)$ *is upper-bounded as:*

$$\Delta^*(r) \leq \text{lowc}\left(\Delta_{\text{Ub}}(r)\right).$$

with

$$\Delta_{\text{Ub}}(r) \triangleq \begin{cases} \min_{i\in\{1,2\}} \Delta_{\text{Ub},i}(r) & \text{if } r < K/2 \\ \frac{1}{K}\left(1-\frac{r}{K}\right) & \text{if } r \geq K/2 \end{cases}.$$

For fixed K, define for each $r \in [\lceil K/2 \rceil - 1]$:

$$\Delta_{\text{Ub},1}(r) \triangleq \left(1-\frac{r}{K}\right) \cdot \frac{r(K-1)+K-r-1}{r(K-1)^2+r(K-2)}.$$

For $K = 5$ and $r = 2$, define

$$\Delta_{\text{Ub},2}(r) \triangleq \left(1-\frac{r}{K}\right) \cdot \frac{7}{30}$$

and for all odd values $K \geq 7$ and $r = (K-1)/2$, set:

$$\Delta_{\text{Ub},2}(r) \triangleq \frac{1}{K}\left(1-\frac{r}{K}\right)\left(1+\frac{1}{(K-r-1)(K-1)}\right).$$

For all other values of r and K, set $\Delta_{\text{Ub},2}(r) = \infty$.

Proof of Lower Bound

Lemma

Consider two disjoint sets $\mathcal{T}$ and $\mathcal{R}$ of same size, and $\mathcal{F} \triangleq [\mathrm{K}] \backslash (\mathcal{R} \cup \mathcal{T})$. Let $\mathcal{M} \subseteq [\mathrm{N}]$ be the set of files known only to nodes $\mathcal{T}$ but not to any other node and partition the set of all IVAs $\mathcal{A}$ it into the following disjoint subsets:

$$\mathcal{W}_r \triangleq \{a_{j,m}\}_{\substack{j \in \mathcal{R} \\ m \in [\mathrm{N}] \backslash \mathcal{M}_j}}, \quad \mathcal{W}_t \triangleq \{a_{j,m}\}_{\substack{j \in \mathcal{T} \\ m \in \mathcal{M} \backslash \mathcal{M}_j}}.$$

For any sequence of distributed computing systems:

$$d \triangleq \overline{\lim_{P \to \infty}} \; \overline{\lim_{\mathrm{T} \to \infty}} \frac{\mathrm{A}}{\mathrm{T} \log \mathrm{P}} \leq \frac{|\mathcal{T}|}{|\mathcal{W}_t| + |\mathcal{W}_r|}$$

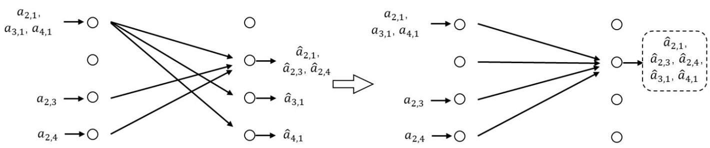


Figure: Example of Lemma with $\mathrm{K} = 4$ and $r = 1$.

- The lower bound can be obtained by linear combination of the Lemma with different $(\mathcal{T}, \mathcal{R})$

Lower Bound for Wireless DC System

Theorem

The computation-NDT tradeoff $\Delta^*(\mathsf{r})$ is lower-bounded as:

$$\Delta_{\mathsf{Lb}}(\mathsf{r}) \leq \Delta^*(\mathsf{r}).$$

$$\Delta_{\mathsf{Lb}}(\mathsf{r}) \triangleq \begin{cases} \frac{1}{\mathsf{K}}\left(2-\frac{3}{\mathsf{K}}\right) & \text{if } \mathsf{r}=1, \\ \frac{1}{\mathsf{K}}\left(1-\frac{\mathsf{r}}{\mathsf{K}}+\max_{t\in[\lfloor \mathsf{K}/2\rfloor]} \text{lowc}\,(C_t(\mathsf{r}))\right) & \text{if } \mathsf{r}\in\left(1,\lceil\frac{\mathsf{K}}{2}\rceil\right), \\ \frac{1}{\mathsf{K}}\left(1-\frac{\mathsf{r}}{\mathsf{K}}\right) & \text{if } \mathsf{r}\in\left[\lceil\frac{\mathsf{K}}{2}\rceil,\mathsf{K}\right], \end{cases}$$

where for any $t \in [\lfloor \mathsf{K}/2 \rfloor]$:

$$C_t(i) = \frac{\binom{\mathsf{K}-i}{t-i}}{\binom{\mathsf{K}}{t}\cdot t}\cdot(\mathsf{K}-2t),$$

Numerical Results

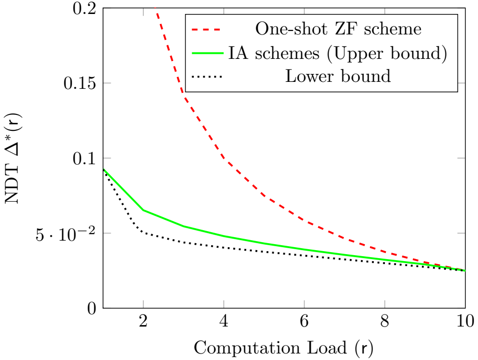
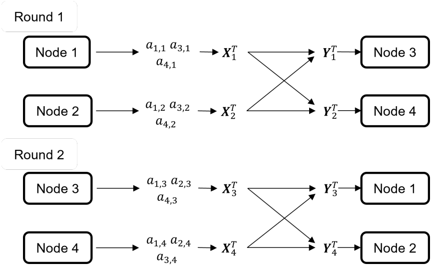


Figure: Lower bound and upper bounds on $\Delta^*(r)$ when $K = 20$.

Wireless DC System with Half-duplex Nodes

- We compare the bounds for wireless DC system with full-duplex nodes and half-duplex nodes.



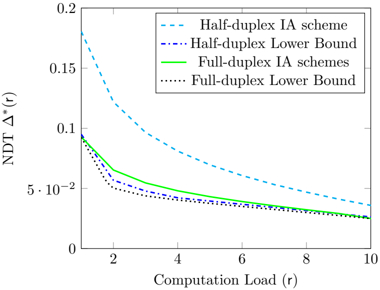


Figure: Bounds on $\Delta^*(r)$ when K = 20.

Summary and Outlook of Part II

- Revealed parallels between wireless DC and partially connected network.
- Achieved near-optimal communication-computation tradeoff.
- Developed information-theoretic converse bound for both full-duplex and half-duplex systems.

Outlook:

- Analyze how input file assignment affects system performance.
- Apply to specific functions: binary, linear separable, etc.
- Practical implementation: finite block length, imperfect CSI, etc.

THANK YOU!

Journal papers:

- **Y. Bi**, M. Wigger, Y. Wu, "Normalized Delivery Time of Wireless MapReduce", in *IEEE Transaction on Information Theory*, vol. 70, no. 10, pp. 7005-7022, Oct. 2024, doi: 10.1109/TIT.2024.3423710.
- Z. Huang, K. Yuan, S. Ma, **Y. Bi** and Y. Wu, "Coded Computing for Half-Duplex Wireless Distributed Computing Systems via Interference Alignment," in *IEEE Transactions on Wireless Communications*, vol. 23, no. 11, pp. 17399-17414, Nov. 2024, doi: 10.1109/TWC.2024.3453403.

Conferences:

- **Y. Bi**, P. Ciblat, M. Wigger, and Y. Wu, "DoF of a Cooperative X-Channel with an Application to Distributed Computing," in *2022 IEEE International Symposium on Information Theory (ISIT)*, Jun. 2022.
- **Y. Bi**, M. Wigger, Y. Wu, "A New Interference-Alignment Scheme for Wireless MapReduce", in *2023 IEEE Globecom*, Dec. 2023.
- **Y. Bi**, Y. Wu, C. Hua, "DoF Analysis for (M, N)-Channels through a Number-Filling Puzzle",in *2024 IEEE ISIT*, Jul. 2024.

- $\mathcal{H}$: Set of all channel coefficients
- $\mathcal{W}_t$, $\mathcal{W}_r$: Target and recoverable IVAs
- $\mathcal{W}_c \triangleq \mathcal{A} \setminus (\mathcal{W}_t \cup \mathcal{W}_r)$
- $\mathbf{Y}_{\mathcal{A}} \triangleq [\mathbf{Y}_j]_{j \in \mathcal{A}}$

- Entropy Decomposition

$$
\begin{aligned}
H(\mathcal{W}_t, \mathcal{W}_r) &= H(\mathcal{W}_t, \mathcal{W}_r | \mathcal{W}_c, \mathcal{H}) \\
&= I(\mathcal{W}_t, \mathcal{W}_r; \mathbf{Y}_{\mathcal{R}} | \mathcal{W}_c, \mathcal{H}) + H(\mathcal{W}_t, \mathcal{W}_r | \mathbf{Y}_{\mathcal{R}}, \mathcal{W}_c, \mathcal{H}) \\
&\leq h(\mathbf{Y}_{\mathcal{R}} | \mathcal{W}_c, \mathcal{H}) - h(\mathbf{Z}_{\mathcal{R}}) + T\epsilon_T + H(\mathcal{W}_t | \mathcal{W}_r, \mathcal{W}_c, \mathbf{Y}_{\mathcal{R}}, \mathcal{H})
\end{aligned}
$$

Bounding Remaining Entropy

- By Fano's inequality

$$
\begin{aligned}
H(\mathcal{W}_t|\mathcal{W}_r, \mathcal{W}_c, \mathbf{Y}_{\mathcal{R}}, \mathcal{H}) &\le I(\mathcal{W}_t; \mathbf{Y}_{\mathcal{F}}|\mathcal{W}_r, \mathcal{W}_c, \mathbf{Y}_{\mathcal{R}}, \mathcal{H}) + T\epsilon'_T \\
&= h(\bar{\mathbf{Y}}_{\mathcal{F}}|\bar{\mathbf{Y}}_{\mathcal{R}}, \mathcal{H}) - h(\mathbf{Z}_{\mathcal{F}}) + T\epsilon'_T
\end{aligned}
$$

  - $\bar{\mathbf{Y}}_j \triangleq \mathbf{H}_{j,\mathcal{T}}\mathbf{X}_{\mathcal{T}} + \mathbf{Z}_j$
  - If $\mathbf{H}_{\mathcal{R},\mathcal{T}}$ is invertible:

- Cleaned Signals and Residual Noise

$$
\begin{aligned}
h(\bar{\mathbf{Y}}_{\mathcal{F}}|\bar{\mathbf{Y}}_{\mathcal{R}}, \mathcal{H}) &\le h(\bar{\mathbf{Z}}_{\mathcal{F}}) + \mathbb{P}(E=0)h(\bar{\mathbf{Y}}_{\mathcal{F}}|\bar{\mathbf{Y}}_{\mathcal{R}}, \mathcal{H}, E=0) \\
&= h(\bar{\mathbf{Z}}_{\mathcal{F}}) \text{ since } \mathbb{P}(E=0) = 0
\end{aligned}
$$

- Final Bound and Conclusion

$$
\begin{aligned}
H(\mathcal{W}_t, \mathcal{W}_r) &\le h(\mathbf{Y}_{\mathcal{R}}|\mathcal{H}) - h(\mathbf{Z}_{\mathcal{R}}) + h(\bar{\mathbf{Z}}_{\mathcal{F}}) - h(\mathbf{Z}_{\mathcal{F}}) + T(\epsilon_T + \epsilon'_T) \\
&\le T|\mathcal{R}|\log(P) + TC_{T,\mathcal{H}} \\
\Rightarrow \frac{|\mathcal{W}_t| + |\mathcal{W}_r|}{T\log P} &\le |\mathcal{R}| = |\mathcal{T}|, \text{ as } P \to \infty
\end{aligned}
$$