Channel Capacity under the Subtractive Dithered Quantization Model

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Introduction

Modern communications rely on **analog-to-digital converters (ADCs)**. Low-precision ADCs (1–3 bits/sample) reduce power and hardware complexity, especially in mmWave and massive MIMO systems. Closed-form capacity expressions for AWGN channels with coarse quantization are generally intractable.

- For AWGN channels under peak and average power constraints, the capacity-achieving input is **discrete with finite support** [2].
- For a K-level quantizer, at most $K+1$ symmetric mass points suffice [1]; we observe numerically that $N=K$ already approaches capacity.

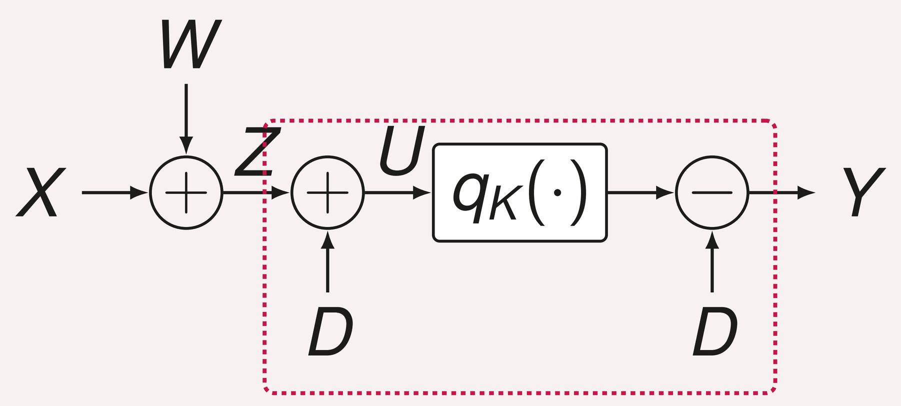
Contributions

We study an AWGN channel followed by a *subtractive dithered* K-level quantizer. By the dithering principle [3], the effective noise is **Gaussian** + **Uniform**, independent of the input.

- We derive **upper and lower bounds on capacity** under peak-amplitude and average-power constraints.
- The **variance-based UB** and **EPI LB** are **tight at moderate SNR**, providing simple closed-form approximations.
- $N=K$ **mass points** achieve near-optimal rates; at **low SNR**, $N=2$ points suffice for any K.
- **Future:** tighter UBs via duality methods; prove $N \leq K$ optimality under the dithered model.

System Model

Channel: $Z = X + W$, $W \sim \mathcal{N}(0, \sigma^2)$, independent of X.
Quantizer: K-level, range $[-\gamma, \gamma]$, step $\Delta = 2\gamma/K$, dither $D \sim \mathcal{U}(-\Delta/2, \Delta/2)$.



Subtractive Dithered Quantizer

Under the Schuchman conditions [3] with negligible overload, the channel reduces to the additive model:

$$Y = X + \overline{W}, \quad \overline{W} = W + W_q, \quad W_q \sim \mathcal{U}(-\tfrac{\Delta}{2}, \tfrac{\Delta}{2})$$

where $\overline{W}$ is **independent of** X.
Input constraints from $\Pr(|U| > \gamma \,|\, X = x) \leq \varepsilon$ on the support of P_X:

- **Peak amplitude:** $|X| \leq A$, where A solves
$$1 - [F_{\overline{W}}(\gamma - A) - F_{\overline{W}}(-\gamma - A)] = \varepsilon.$$
- **Average power:** $\mathbb{E}[X^2] \leq P_0$.
$$C(A, P_0) = \sup_{P_X \in \mathcal{F}} \left[h(Y) - h(\overline{W})\right].$$

Capacity Bounds

Theorem 1 — Variance-Based Upper Bound

$$C \leq \tfrac{1}{2}\log\left(2\pi e\left(\min\{A^2, P_0\} + \sigma^2 + \tfrac{\Delta^2}{12}\right)\right) - h(\overline{W})$$

Gaussian maximizes entropy under a fixed variance.
Asymptotics.
- $P_0 \to 0$: bound $\to \frac{1}{2}\log(2\pi e(\sigma^2 + \Delta^2/12)) - h(\overline{W}) > 0$ **(loose)**.
- $P_0 \to \infty$: saturates at $\frac{1}{2}\log(2\pi e(A^2 + \sigma^2 + \Delta^2/12)) - h(\overline{W})$ — **capacity is finite**.

Theorem 2 — EPI-Based Lower Bound

$$C \geq \tfrac{1}{2}\log\left(e^{2h_{\max}(A,P_0)} + e^{2h(\overline{W})}\right) - h(\overline{W})$$

$h_{\max}$ is achieved by a **truncated Gaussian** $f_X^\star \propto e^{-\mu^\star x^2}$, $|x| \leq A$:

$$h_{\max} = \begin{cases} \log(2A), & P_0 \geq A^2/3, \\ -\log c + \mu^\star P_0, & P_0 < A^2/3, \end{cases}$$

where $c = \sqrt{\mu^\star} / \left[\sqrt{\pi}(1 - 2Q(\sqrt{2\mu^\star}A))\right]$, and $\mu^\star > 0$ solves

$$\frac{1}{2\mu} - \frac{A\,e^{-\mu A^2}}{\sqrt{\pi\mu}\,(1 - 2Q(\sqrt{2\mu}\,A))} = P_0.$$

Asymptotics.
- $P_0 \to 0$: $\mu^\star \sim 1/(2P_0)$, so $C \gtrsim \frac{1}{2}\log(1 + 2\pi e P_0 e^{-2h(\overline{W})})$ **(linear in P_0)**.
- $P_0 \geq A^2/3$: saturates at $\frac{1}{2}\log(4A^2 + e^{2h(\overline{W})}) - h(\overline{W})$ **(below the UB saturation)**.

Discrete-Input Lower Bounds. Symmetric N-point constellations with $I(X;Y)$ maximized numerically over positions and probabilities.

Numerical Results

$(\sigma^2 = 10^{-2},\ \varepsilon = 10^{-4},\ \mathrm{SNR} = P_0/\sigma^2)$

$K=4$ **levels**, $\gamma=2$, $A=1.228$ **(low-SNR & high-SNR regimes)**

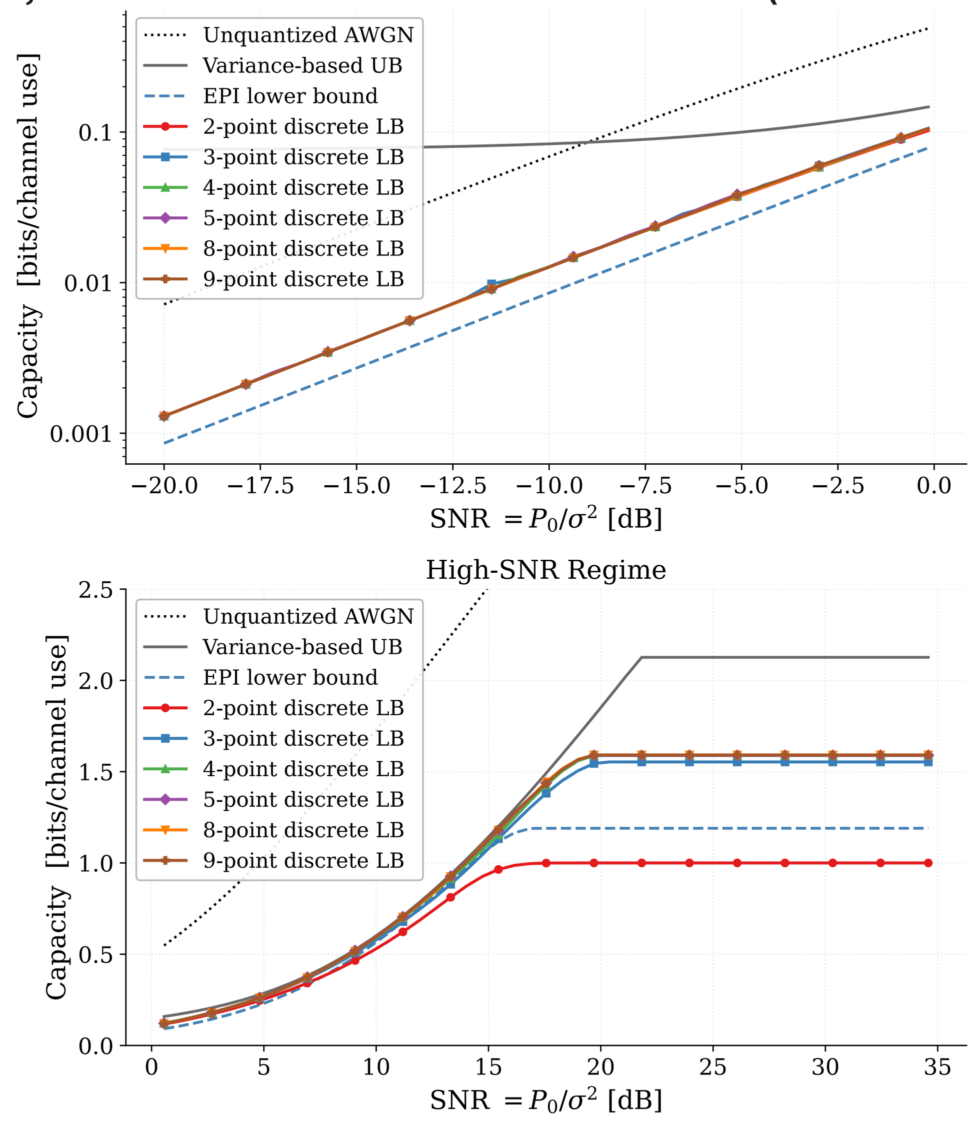


Fig. 1: Capacity bounds for $K=4$ quantization levels.

$K=8$ **levels — effect of dynamic range** γ **($\gamma=2$ vs. $\gamma=5$)**

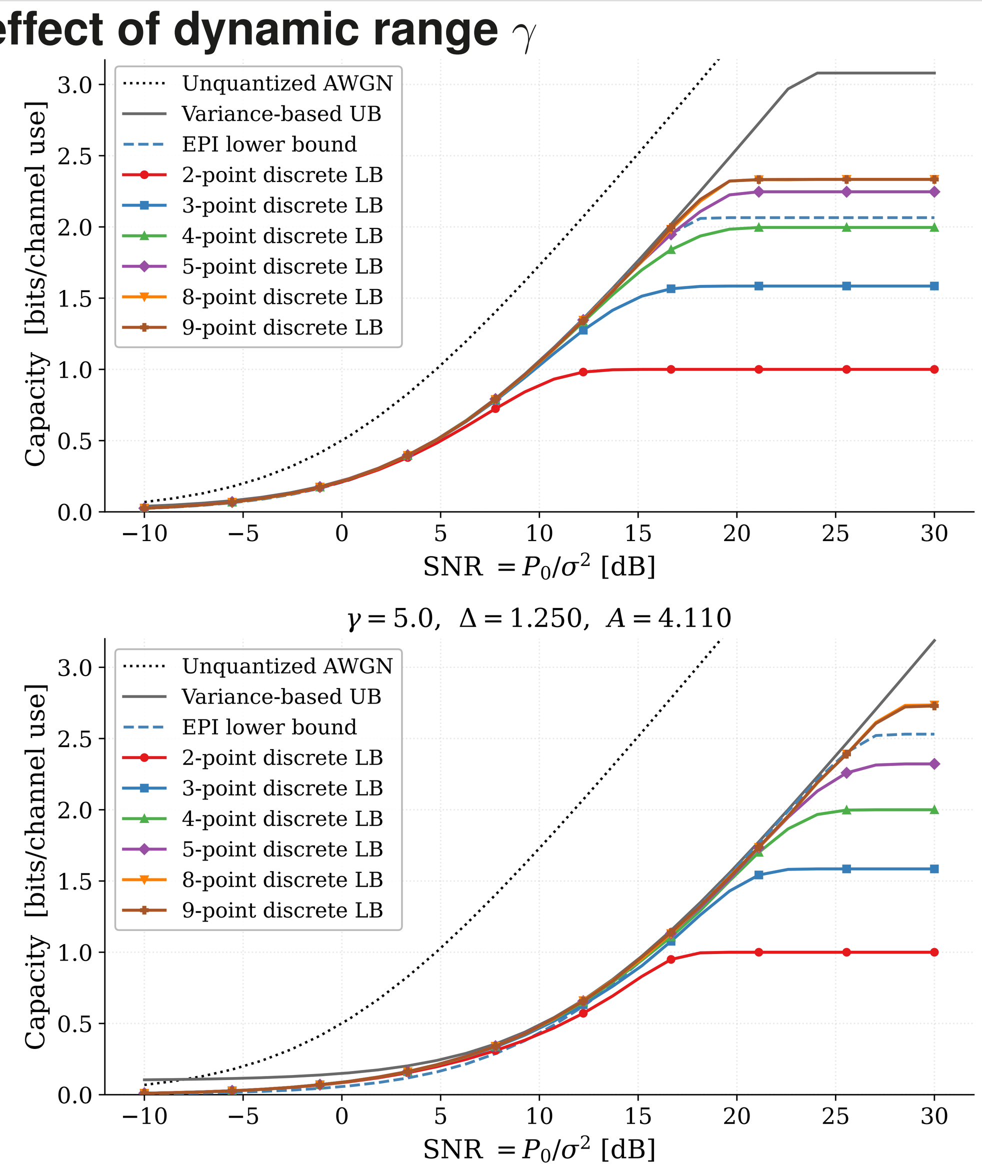


Fig. 2: Capacity bounds for $K=8$ levels with two dynamic ranges $\gamma \in \{2, 5\}$.

Comparison with sign quantizer [1] **($K=2$, varying γ)**

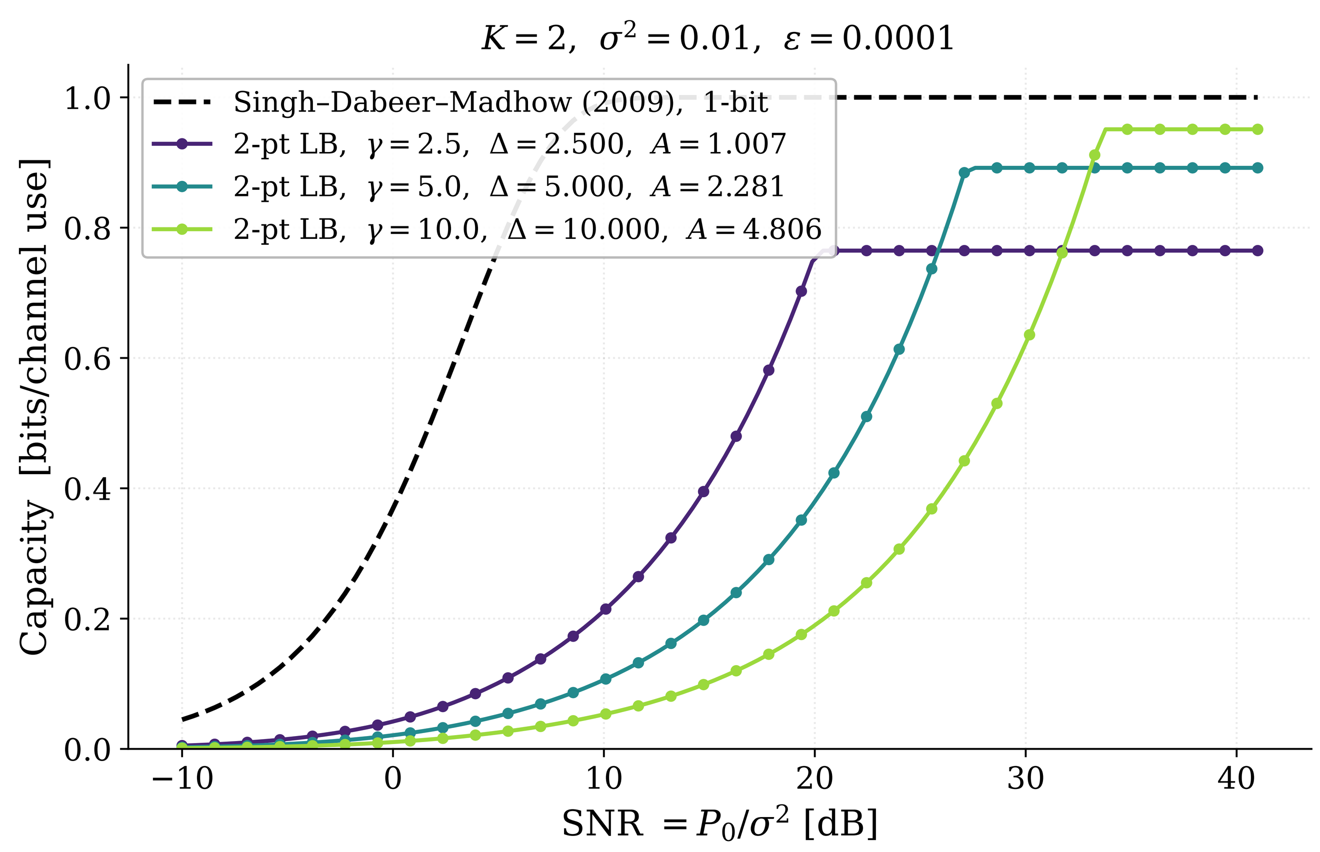


Fig. 3: 1-bit sign-quantizer bounds of Singh et al. [1] vs. dithered model for varying γ.

Dithering introduces uniform noise absent in the sign-quantizer model, creating a **large capacity gap at low SNR**. As γ increases, A relaxes the peak constraint and the gap narrows at high SNR.

References

[1] J. Singh, O. Dabeer, and U. Madhow, "On the limits of communication with low-precision analog-to-digital conversion at the receiver," *IEEE Trans. Commun.*, vol. 57, no. 12, pp. 3629–3639, Dec. 2009.
[2] J. G. Smith, "The information capacity of amplitude- and variance-constrained scalar Gaussian channels," *Inf. Control*, vol. 18, no. 3, pp. 203–219, Apr. 1971.
[3] R. M. Gray and T. G. Stockham, "Dithered quantizers," *IEEE Trans. Inf. Theory*, vol. 39, no. 3, pp. 805–812, May 1993.

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