Zero-Rate Independence Testing

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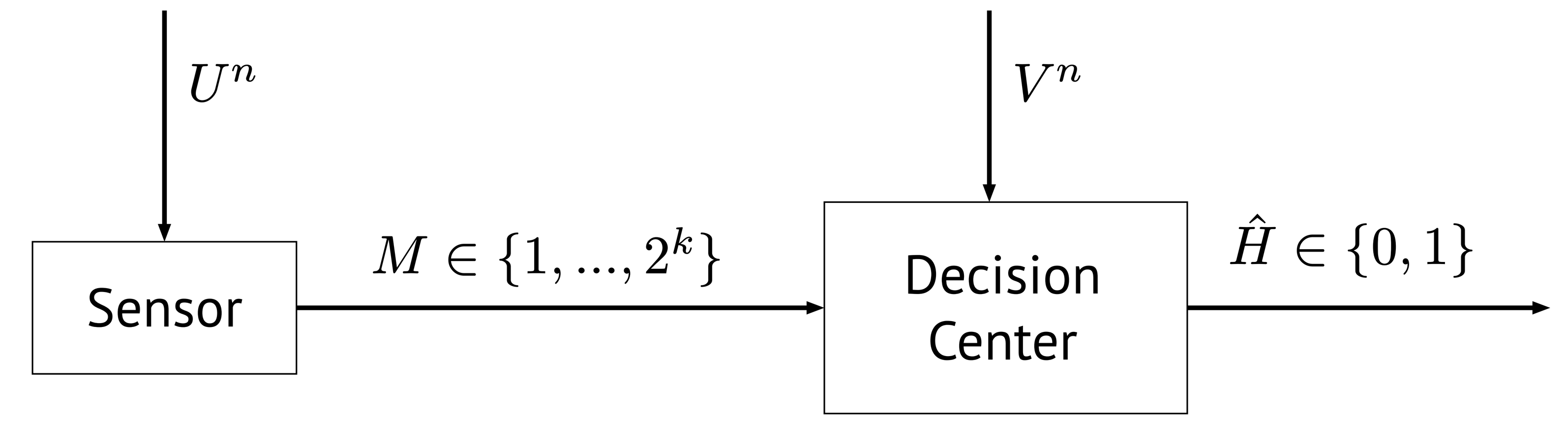

Figure 1: Hypothesis Testing with One-Sided Compression Constraint.

Distributed Hypothesis Testing

Samples (U^n, V^n) are generated according to either

$$H = 0 : (U^n, V^n) \overset{\text{IID}}{\sim} P_{UV} \quad \textbf{or} \quad H = 1 : (U^n, V^n) \overset{\text{IID}}{\sim} Q_{UV}$$

for P_{UV} and Q_{UV} probability distributions over a *finite* alphabet with $P_{UV} \ll Q_{UV}$.

Decision center tries to guess the underlying hypothesis H, helped by a **sensor** connected via a *noiseless zero-rate* link, so that $k(n)$ bits are transmitted with

$$\lim_{n\to\infty} \frac{k(n)}{n} = 0.$$

Sensor
Observes U^n
Sends $k(n)$ bits message $M = f_n(U^n)$

Decision Center
Observes V^n and obtains M
Takes decision $\hat{H} = g_n(M, V^n)$

Error Criteria and Goal

A pair of encoding and test (f_n, g_n) for blocklength n achieves errors

$$\alpha_n := \mathbb{P}\left[\hat{H} = 1 | H = 0\right] \quad \text{and} \quad \beta_n := \mathbb{P}\left[\hat{H} = 0 | H = 1\right]$$

Let β_n^ε be the smallest achievable β_n when imposing a threshold $\varepsilon \in (0, 1)$ on α_n

$$\beta_n^\varepsilon = \inf\{\beta_n : (f_n, g_n) \text{ such that } \alpha_n \leq \varepsilon\}.$$

Stein's regime: objective is to characterize the behavior of β_n^ε as $n \to \infty$.

Han-Papamarcou-Shalaby Exponent

β_n^ε decays exponentially with n as

$$\beta_n^\varepsilon = 2^{-n\cdot\theta^* + o(n)}, \text{ that is } \lim_{n\to\infty} -\frac{1}{n}\log\beta_n^\varepsilon = \theta^*,$$

where $\theta^* := \min\{\mathrm{D}(P_{UV}^* \| Q_{UV}) : P_{UV}^*$ such that $P_U^* = P_U$ and $P_V^* = P_V\}$.

The Han-Papamarcou-Shalaby exponent θ^* does *not* depend on ε neither on k.
It is achieved using a single bit of communication.

Testing Against Independence

Under $H = 1$, U^n and V^n are *independent* as $Q_{UV} = P_U P_V$, with same marginal distributions as under $H = 0$.

The Han-Papamarcou-Shalaby exponent is *vacuous:* $\theta^* = 0$ (take $P_{UV}^* = Q_{UV}$).
β_n^ε cannot decrease exponentially with n, but only slower.

What is the asymptotic behavior of β_n^ε when
testing against independence with zero-rate communication?

Bounded Message Size

Number of transmitted bits is bounded

$$\lim_{n\to\infty} k(n) < \infty$$

Impossibility

β_n^ε cannot vanish when the message size is bounded

$$\varliminf_{n\to\infty} \beta_n^\varepsilon > 0.$$

$\Rightarrow$ Message size is the limiting parameter for independence testing.

Unbounded Message Size

Number of transmitted bits grows sublinearly with blocklength n with

$$\lim_{n\to\infty} k(n) = \infty$$

Naive Scheme

Sensor transmits the first k bits of U^n, for $\tau := k/\,\mathrm{H}(U)$:

Sensor
Sends message $M = U^\tau$

Decision Center
Performs a local test with (U^τ, V^τ)

This achieves vanishing α_n and exponential decay of β_n in k, so that

$$\varliminf_{n\to\infty} -\frac{1}{k}\log\beta_n^\varepsilon \geq \frac{\mathrm{I}(U;V)}{\mathrm{H}(U)}.$$

Achievability

Using Han's scheme, we show that

$$\varliminf_{n\to\infty} -\frac{1}{k}\log\beta_n^\varepsilon \geq \eta(P_U, P_{V|U})$$

where $\eta(P_U, P_{V|U}) \in [0, 1]$ is the *contraction coefficient* of channel $P_{V|U}$ with input distribution P_U, defined as

$$\eta(P_U, P_{V|U}) := \sup_{\substack{P_{W|U} \\ W - U - V}} \frac{\mathrm{I}(W;V)}{\mathrm{I}(W;U)}.$$

This outperforms the naive scheme: $\eta(P_U, P_{V|U})$ is in general not achieved by taking $W = U$.

Converse

We have two partial results:

$$\textbf{Weak Converse:} \ \varlimsup_{n\to\infty} -\frac{1}{k}\log\beta_n^\varepsilon \leq \frac{\eta(P_U, P_{V|U})}{1 - \varepsilon},$$

which gives a proper converse in the limit $\varepsilon \to 0$,

$$\textbf{Strong Converse:} \ \varlimsup_{n\to\infty} -\frac{1}{k}\log\beta_n^\varepsilon \leq \eta(P_U, P_{V|U}) \text{ when } \lim_{n\to\infty}\frac{k(n)}{\sqrt{n}} = \infty, \quad (1)$$

yielding a converse for all ε provided $k(n)$ grows sufficiently fast as in (1).

$\Rightarrow \beta_n^\varepsilon$ decays exponentially with $k(n)$. Under (1) we get

$$\beta_n^\varepsilon = 2^{-k\cdot\eta(P_U, P_{V|U}) + o(k)},$$

whereas in general a $(1 - \varepsilon)^{-1}$ gap between exponents remains.

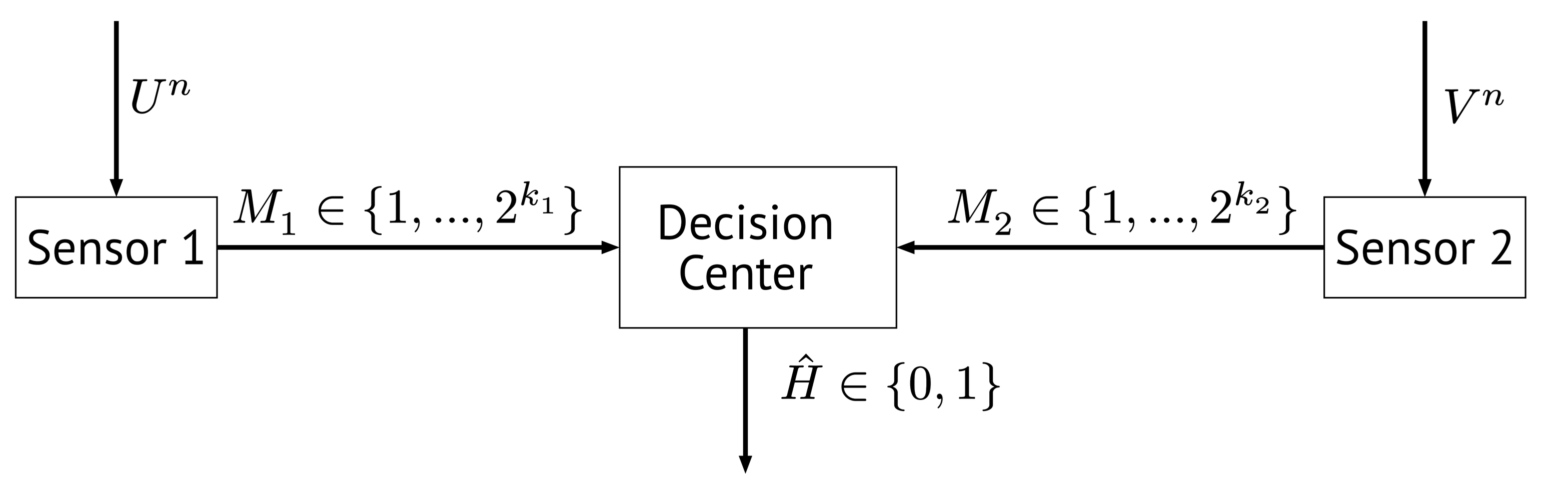

Figure 2: Hypothesis Testing with Two-Sided Compression Constraint.

Two-Sided Compression

Both U^n and V^n are observed remotely via *noiseless zero-rate* links.
The Han-Papamarcou-Shalaby exponent still holds.

$\Rightarrow$ Asymptotic of β_n^ε is governed by $\min(k_1, k_2)$:

- For *bounded* $\min(k_1, k_2)$, vanishing error is still impossible
$$\varliminf_{n\to\infty} \beta_n^\varepsilon > 0.$$
- For *unbounded* $\min(k_1, k_2)$, β_n^ε decays exponentially with $\min(k_1, k_2)$.

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