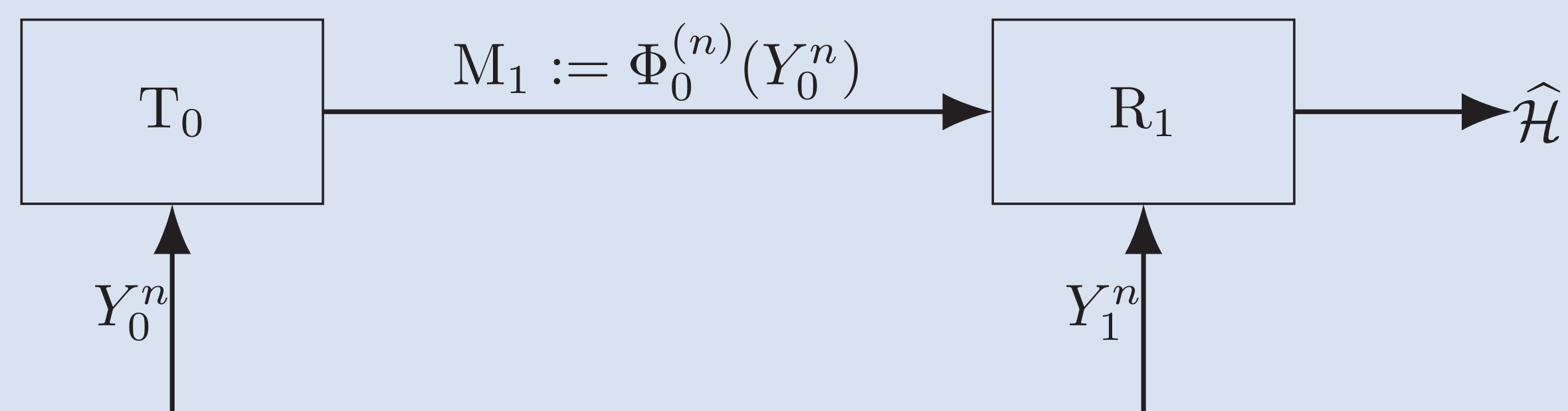


- **Objective:** Provide empirical results of the probabilities of false alarm and missed detection and the error exponent.
- **Contribution:** Monte Carlo implementation of single-hop hypothesis testing over BSC, simulating sequence generation, encoding, and two-stage typicality tests for transmission and decision. Empirical false alarm and missed detection probabilities are estimated via repeated trials under both hypotheses, for different blocklengths.

System model



The observations (Y_0, Y_1) satisfy the Markov chain $Y_0 \rightarrow Y_1$ and, according to the hypothesis $\mathcal{H} \in \{0, 1\}$, follow:

$$\mathcal{H} = 0 : (Y_0^n, Y_1^n) \stackrel{\text{i.i.d.}}{\sim} P_{Y_0 Y_1}, \quad (1)$$

$$\mathcal{H} = 1 : (Y_0^n, Y_1^n) \stackrel{\text{i.i.d.}}{\sim} P_{Y_0} P_{Y_1}. \quad (2)$$

The type-I and type-II error probabilities are defined by:

$$\alpha_n \triangleq \mathbb{P}[\hat{\mathcal{H}} = 1 \mid \mathcal{H} = 0], \quad (3)$$

$$\beta_n \triangleq \mathbb{P}[\hat{\mathcal{H}} = 0 \mid \mathcal{H} = 1]. \quad (4)$$

The type-II error exponent is:

$$\theta \triangleq \lim_{n \rightarrow \infty} -\frac{1}{n} \log \beta_n. \quad (5)$$

Constraint for type-I error

$$\lim_{n \rightarrow \infty} \alpha_n \leq \epsilon \quad (6)$$

The best achievable error exponent under (6) is:

$$\eta(R) \triangleq \max_{\substack{P_{U|Y_0} \\ I(U; Y_0) \leq R}} I(U; Y_1), \quad (7)$$

With appropriate implementations of T_0 and R_1 one can achieve all

$$\theta \leq \eta(R). \quad (8)$$

Probability distribution for binary symmetric channels (BSC)

Under hypothesis $H = 0$:

$$Y_0^n \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_0), \quad (9)$$

$$U^n = Y_0^n \oplus V^n, \quad V^n \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_v) \quad (10)$$

$$Y_1^n = Y_0^n \oplus Z_1^n, \quad Z_1^n \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_1), \quad (11)$$

where \oplus is the bit wise xor operation and V and Z_1 are independent. U is an auxiliary variable and p_v is determined thanks to R and maximizes the error exponent

Under hypothesis $H = 1$:

$$Y_1^n \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(p_0). \quad (12)$$

Properties

For $R \in [0, 1]$ and under above BSC model, we have:

$$\eta(R) = 1 - H_b(H_b^{-1}(1 - R) \star p_1) \quad (13)$$

$$H_b(x) := -x \log(x) - (1 - x) \log(1 - x), \quad (14)$$

where H_b is the binary entropy, H_b^{-1} is its inverse over the domain $[0, 1/2]$ and \star is the binary convolution:

$$a \star b = a(1 - b) + (1 - a)b \quad \forall (a, b) \in [0, 1]^2. \quad (15)$$

Moreover, we have:

$$p_v = H_b^{-1}(1 - R) \quad (16)$$

Algorithm

Input: n, R, p_0, p_1, K

Output: Empirical probabilities $\hat{\alpha}$ and $\hat{\beta}$

Compute $\mu_n = n^{-1/3}$ and p_v according to (16);

// Counter of the number of false alarm and counter of the number of missed detection

$n_\alpha \leftarrow 0$ and $n_\beta \leftarrow 0$;

for $\mathcal{H} \leftarrow 0$ **to** 1 **do**

for $k \leftarrow 1$ **to** K **do**

 Draw $Y_0^n \sim P_{Y_0}^{\otimes n}$;

 Draw $V^n \sim P_V^{\otimes n}$ and compute $U^n = Y_0^n \oplus V^n$;

 Compute empirical distribution $\hat{P}_{U^n Y_0^n}$;

if $\left\| \hat{P}_{U Y_0} - P_{U Y_0} \right\|_\infty > \mu_n$ **then**

 // If typicality check fails

 Send $M_1 = [0]$;

$\hat{\mathcal{H}} \leftarrow 1$;

if $\mathcal{H} = 0$ **then**

$n_\alpha \leftarrow n_\alpha + 1$;

continue;

else

 Send^a U^n ;

end

 Draw $Y_1^n \sim P_{Y_1|\mathcal{H}}$;

 Compute empirical distribution $\hat{P}_{U^n Y_1^n}$;

if $\left\| \hat{P}_{U^n Y_1^n} - P_{Y_0 Y_1} \right\|_\infty > \mu_n$ **then**

$\hat{\mathcal{H}} \leftarrow 1$;

if $\mathcal{H} = 0$ **then**

$n_\alpha \leftarrow n_\alpha + 1$;

else

$\hat{\mathcal{H}} \leftarrow 0$;

if $\mathcal{H} = 1$ **then**

$n_\beta \leftarrow n_\beta + 1$

end

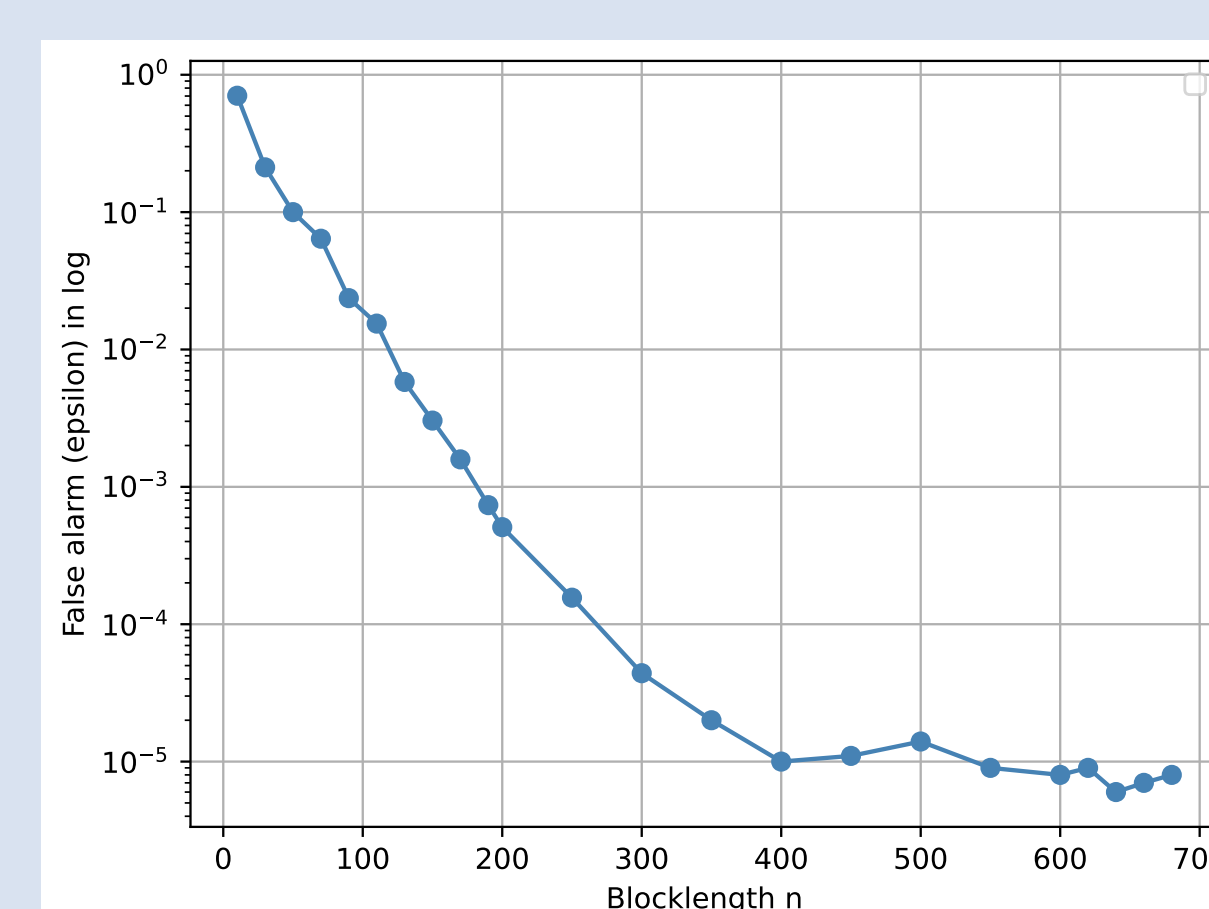
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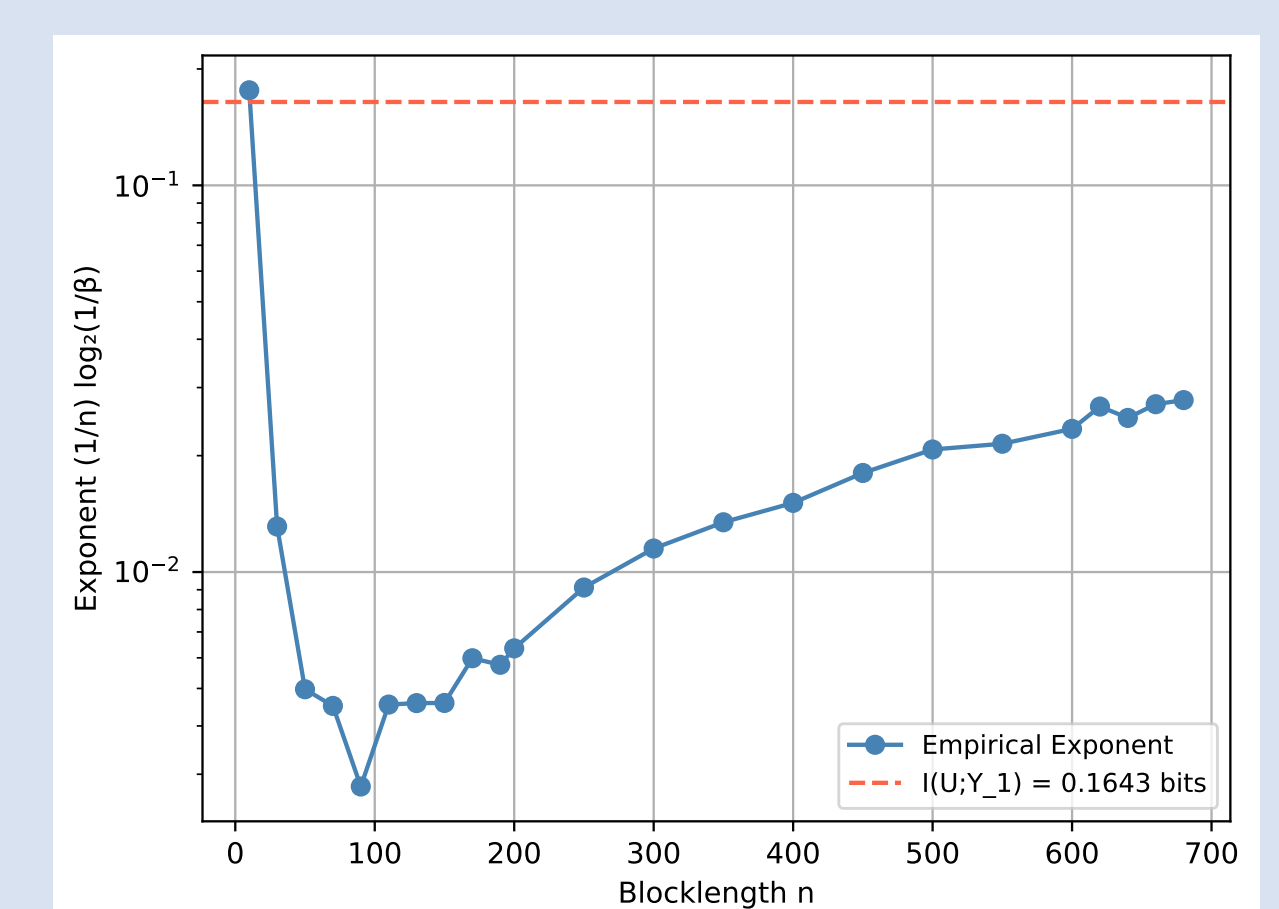
$\hat{\alpha} \leftarrow n_\alpha / K$ and $\hat{\beta} \leftarrow n_\beta / K$;

^aWe assume that U^n is known at both encoder and decoder. In theory, U^n is selected from a codebook $\mathcal{C}_U := \{u^n(m) : m \in \{1, \dots, [2^{nR}]\}\}$ as a sequence jointly typical with Y_0^n . The corresponding index m is then sent. However, explicit codebook handling is infeasible for large n .

Results for $R = 0.5, p_0 = 0.5, p_1 = 0.1$ and $K = 10^6$ iterations



Empirical false-alarm probability vs. n



Empirical error exponent vs. n

Remark and conclusions

- For small blocklengths ($n < 50$), many typicality checks between $\hat{P}_{U Y_0}$ and $P_{U Y_0}$ fail, increasing the empirical error exponent.
- With increasing blocklength, false alarms saturate while missed detections decrease, yielding a higher error exponent.
- **Perspectives:** Extend these simulations in the two-hops case. Use these results for reinforcement learning simulations.