Variable-Length Stop-Feedback Codes

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Outline

Variable-Length Stop-Feedback (VLSF) Codes

Optimization of Sparse VLSF Codes via Saddlepoint Approximation

VLSF Codes over Correlated Noncoherent Fading Channels

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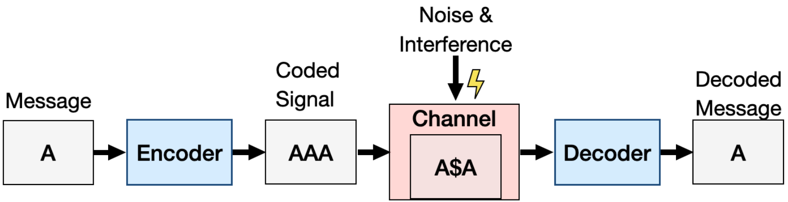

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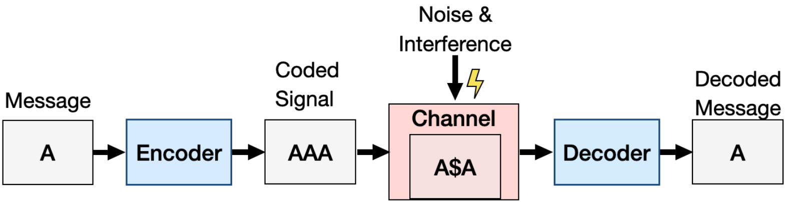

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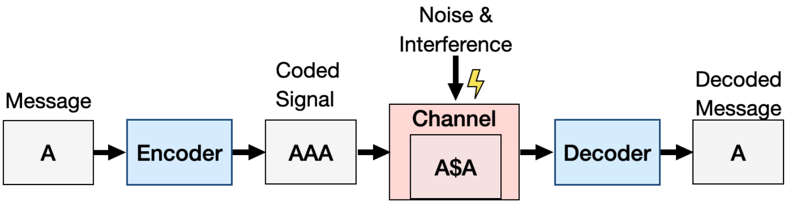

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- LDPC codes: Wi-Fi, Cellular Networks (4G, 5G).
- Turbo codes: used in 3G and 4G systems.

A Tale of Two Conversations

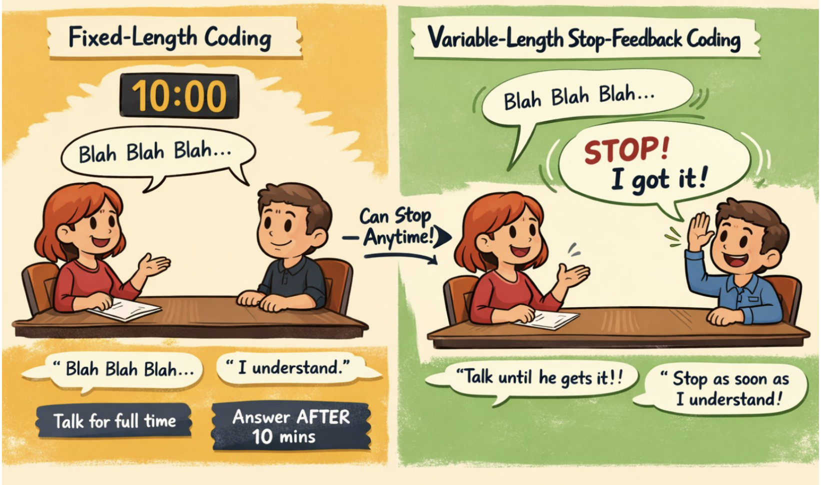

Image generated by AI from my story.

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Definition: VLSF codes

A (M, ϵ) VSLF code achieves a message size M with average error probability $\mathbb{P}(\hat{W} \neq W) \leq \epsilon$.

Hypothesis Testing

Binary Hypothesis Testing

$$(X^n, Y^n) \sim \begin{cases} \mathcal{H}_0 : P_{X^n} P_{Y^n|X^n} & (X^n \text{ was transmitted}) \\ \mathcal{H}_1 : P_{X^n} P_{Y^n} & (X^n \text{ was not transmitted}) \end{cases}$$

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Information density: the log-likelihood ratio:

$$\imath(x^n; y^n) \triangleq \log \frac{\mathrm{d}P_{Y^n|X^n=x^n}}{\mathrm{d}P_{Y^n}}(y^n)$$

where P_{Y^n} is the marginal distribution induced by the input P_X^n.

Information Density Random Walks

Random coding + memoryless i.i.d. channels = random walks

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Characterizing VLSF codes via this **random-walk** structure.

Decoder

At each time n, the decoder tests M codewords by information densities:

$$\{\imath(x_1^n; y^n), \imath(x_2^n; y^n), \cdots, \imath(x_M^n; y^n)\}$$

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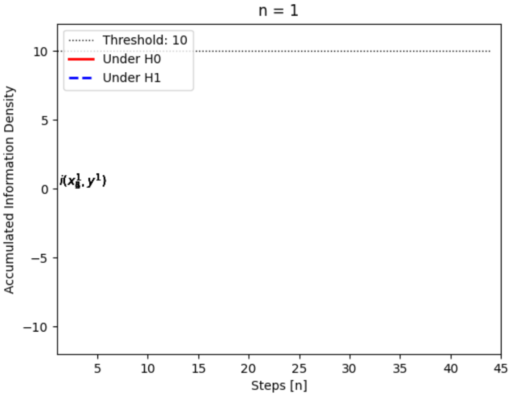

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The VLSF decoder stops at the first time τ where one of these densities exceeds a threshold γ:

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Interpretation

Decoding stops when the target reliability is guaranteed.

Reliability Guarantee

Theorem

Let $\epsilon > 0$ and $M \in \mathbb{N}_{>0}$. For any threshold $\gamma \geq \log \frac{M-1}{\epsilon}$, the threshold-based VLSF decoding rule satisfies

$$\mathbb{P}[W \neq \hat{W}] \leq \epsilon.$$

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5. Choosing $\gamma \geq \log \frac{M-1}{\epsilon}$ yields $\mathbb{P}[W \neq \hat{W}] \leq \epsilon$

$\square$

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Latency:

Stopping time τ:

- τ is upper-bounded by the true message (indexed by 1).

[2] W. Feller, *An introduction to probability theory and its applications, Volume 2*.John Wiley & Sons, 1991.

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Approximation of stopping time distribution[3]

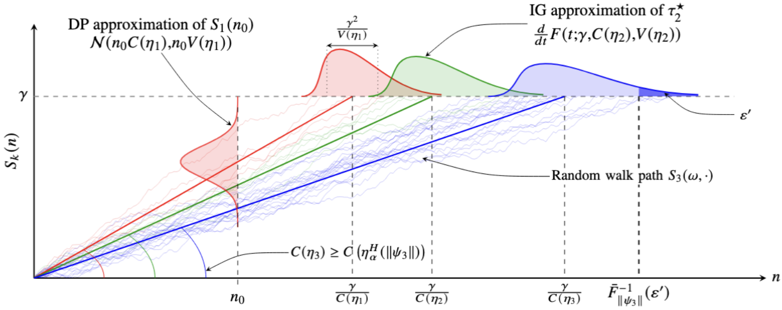

Figure: Diffusion processes (DP) approximation and inverse Gaussian (IG) distributed stopping time under different SNRs

[3] Joint work with Dr. Kevin Zagalo

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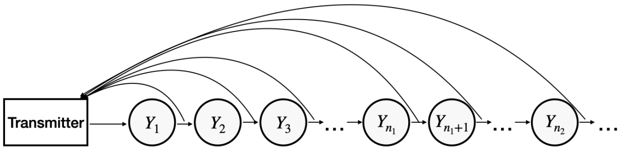

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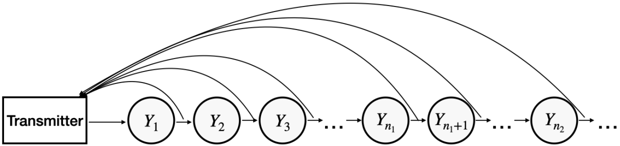

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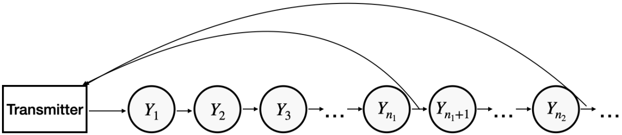

Sparse feedback: Feedback occurs only at t **optimized** time instants: n symbols $=$ t feedback

Optimization problem:

Literature[4]**:** Define t decoding time instants:
$\mathcal{T} \triangleq \{n_1, n_2, \ldots, n_t\}$

[4] H. Yang, R. C. Yavas, V. Kostina, and R. D. Wesel, "Variable-length stop-feedback codes with finite optimal decoding times for BI-AWGN channels," in *2022 ISIT*, pp. 1527–1532

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$$\text{s.t.} \quad (M-1)e^{-\gamma} + \mathbb{P}[\imath(X_1^{n_t}, Y^{n_t}) < \gamma] \leq \epsilon. \tag{1b}$$

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    - At n_t: No crossing: $\mathbb{P}[\imath(X_1^{n_t}, Y^{n_t}) < \gamma]$

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Proposed rule

If no threshold-crossing occurs at n_t, the decoder selects the codeword with the highest information density:

- Error probability characterized by fixed blocklength analysis[5]

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Unknown quantity

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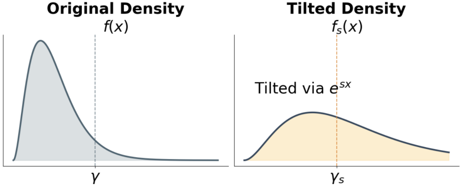
Saddlepoint Approximations

Step 1: $S_n = \sum_{i=1}^{n} \imath(X_{1,i}; Y_i)$ Exponential tilting



$$\boxed{f_s(x) = \frac{f(x)e^{sx}}{M(s)}}$$

$M(s) = \mathbb{E}[e^{s \cdot S_n}]$
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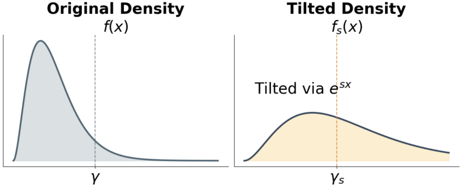

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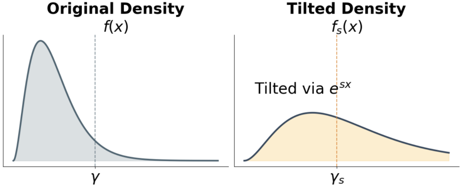

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Step 3: The Lugannani-Rice Approximation

$$\mathbb{P}[S_n \leq \gamma] \approx \Phi(\hat{w}) + \phi(\hat{w})\left(\frac{1}{\hat{w}} - \frac{1}{\hat{u}}\right)$$

$\Phi(\,\cdot\,)$: Standard Normal CDF
$\phi(\,\cdot\,)$: Standard Normal PDF

Definition of terms:

$\hat{w} = \mathrm{sgn}(\hat{s})\sqrt{2(\gamma\hat{s} - nK(\hat{s}))}$:
Signed root of log-likelihood ratio deviation

$\hat{u} = \hat{s}\sqrt{nK''(\hat{s})}$
Related to curvature of tilted distribution

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Results

Optimization problem can be solved analytically.

Accuracy of Saddlepoint Approximations

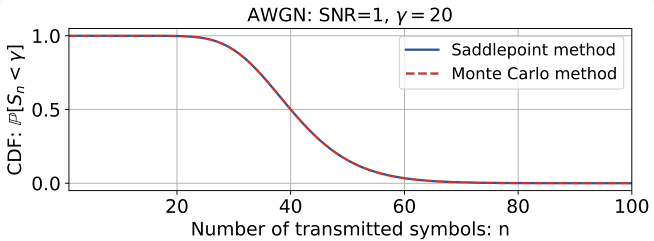

Figure: Saddlepoint approximations for a continuous channel

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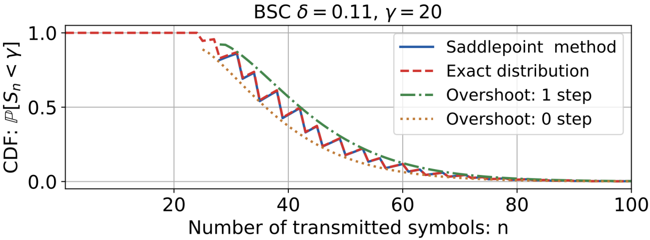
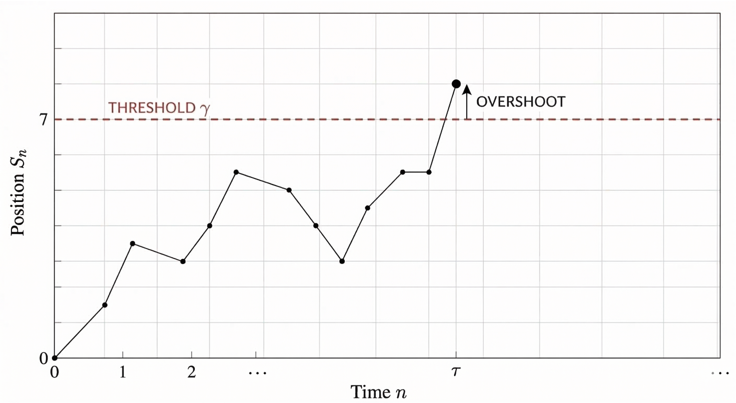


Figure: Saddlepoint approximations for a discrete channel

Random Walk Overshoot



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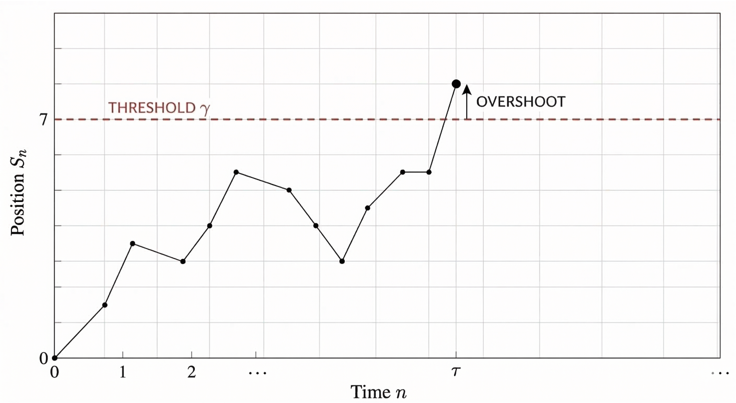

Assuming an overshoot of 0 or 1 step yields continuous correction bounds that can be used for gradient-based optimization.

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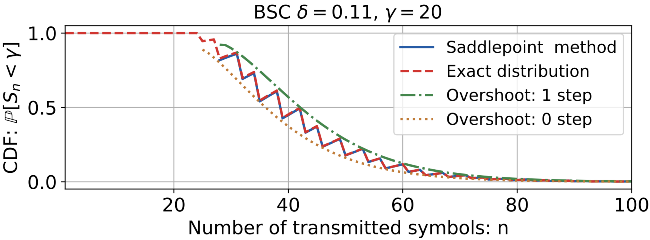

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Literature[6]: No crossing at final decoding: = error

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Results: Number of Feedback and Achievable Rates

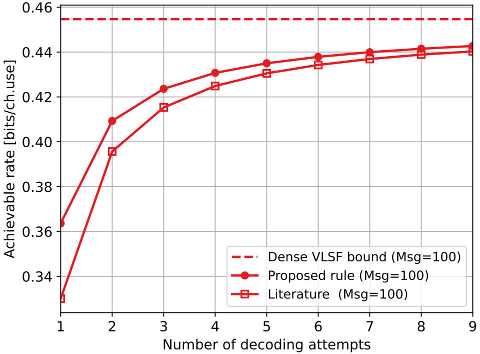

Figure: Achievable rate of sparse VLSF codes versus number of decoding attempts.

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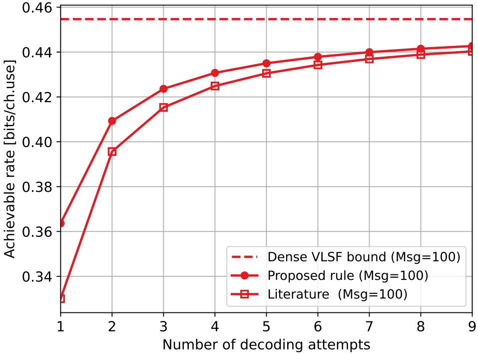

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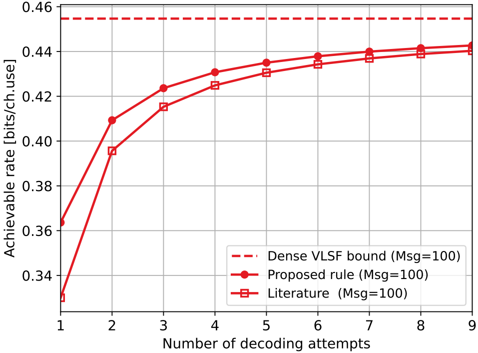

Figure: Achievable rate of sparse VLSF codes versus number of decoding attempts.

- Near-optimal with a small number of decoding attempts.
- Refined decoding rule yields large gains for one or two attempts.

Results: Binary Symmetric Channel

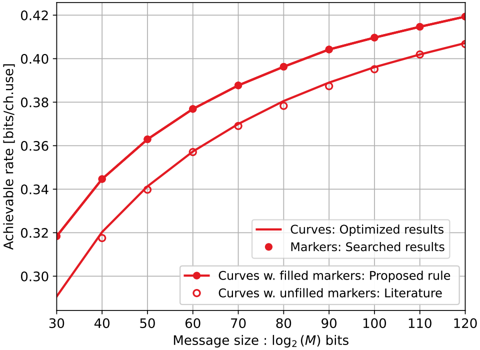

Figure: Achievable rate versus message size (in bits)

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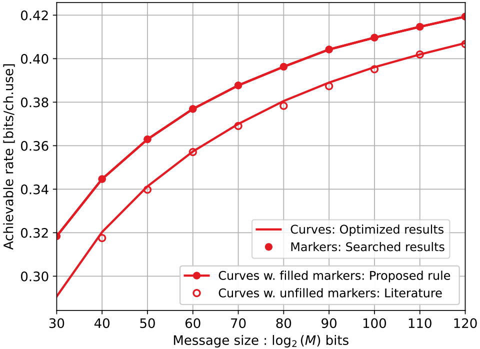

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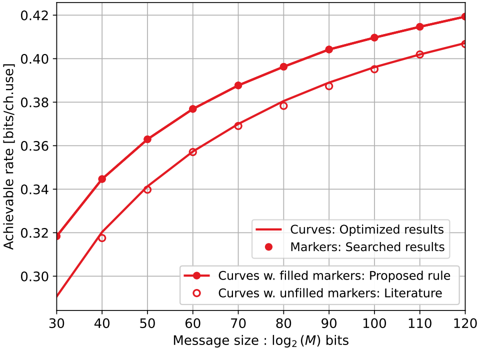

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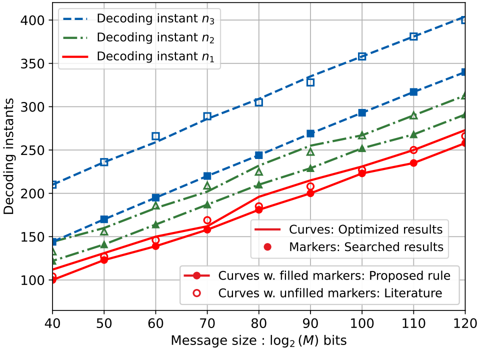

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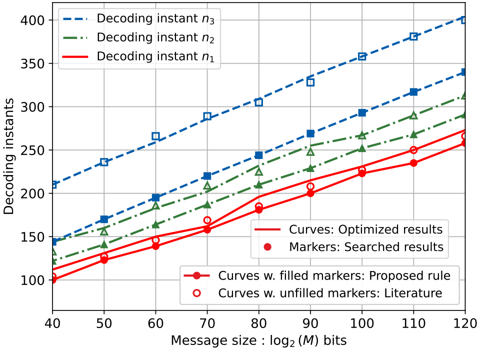


Figure: Optimal decoding instants versus message size (in bits)

- BSC ($p = 0.11$, $t = 3$ attempts)
- Optimized decoding instants occur earlier at later stages ($n2$, $n3$)

Outline

Variable-Length Stop-Feedback (VLSF) Codes

Optimization of Sparse VLSF Codes via Saddlepoint Approximation

VLSF Codes over Correlated Noncoherent Fading Channels

Correlated Noncoherent Fading Channels

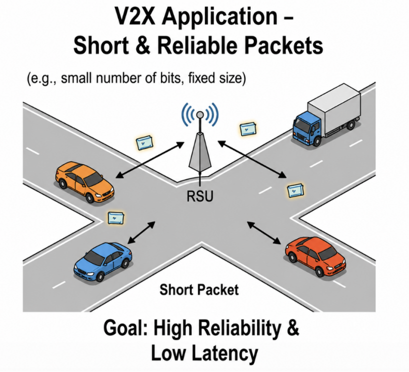
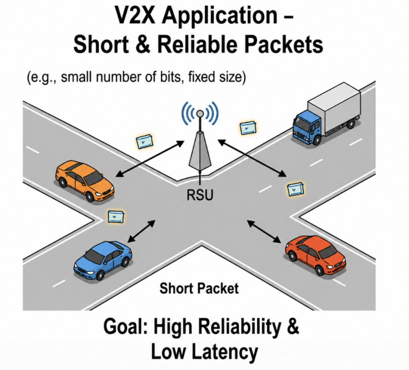


Figure: Vehicular Networks; RSU: Roadside Units

Correlated Noncoherent Fading Channels



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Model Dilemma:

- Fast fading: dynamic channel over codeword duration
- Pilots: channel outdated quickly
- Correlated Noncoherent Fading Channels

Correlated Noncoherent Fading Channels

"Middle Ground" regime

Channels experience symbol-wise correlation, but not block-fading.
Fading channel with memory[7]

[7] Lapidoth, Amos, and Stefan M. Moser. "Capacity bounds via duality with applications to multiple-antenna systems on flat-fading channels." IEEE Transactions on Information Theory 49.10 (2003): 2426-2467.

Correlated Noncoherent Fading Channels

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Example: First-order Gauss-Markov process with correlation ρ

$$H_k = \rho H_{k-1} + \sqrt{1-\rho^2} W_k,$$

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Analysis based on martingale for any nontransmitted codeword is still valid!

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Conditional expectation with respect to $\mathcal{F}_{n-1}$ under $\mathcal{H}_1$:

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\begin{aligned}
&\mathbb{E}_{\mathcal{H}_1}[\mathsf{L}_n(X^n; Y^n)|\mathcal{F}_{n-1}] \\
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=&\mathsf{L}_{n-1}(X^{n-1}; Y^{n-1})\int \frac{\mathrm{d}P_{Y_n|X_n,Y^{n-1}}}{\mathrm{d}P_{Y_n|Y^{n-1}}}(Y_n)\mathrm{d}P_{Y_n|Y^{n-1}}(Y_n) \\
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\end{aligned}
$$

Takeaway

VLSF Decoding rule still holds:

- Decoder declares the message for which the information density crosses the threshold.
- Reliability Guarantee still holds.

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For any measurable set $A \in \mathcal{B}(\mathcal{Y}^n)$,

$$\begin{aligned} P_{Y^n}(A) &= \int\int P_{Y^n|X^n=x^n,H^n=h^n}(A)\mathrm{d}P_{X^n}(x^n)\mathrm{d}P_{H^n}(h^n) \\ &= \int P_{Y^n|H^n=h^n}(A)\mathrm{d}P_{H^n}(h^n). \end{aligned}$$

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$$= \int P_{Y^n|H^n=h^n}(A) \mathrm{d}P_{H^n}(h^n).$$

Solutions

Change-of-measure + Variation approach

$$L(h^n) \triangleq \frac{\mathrm{d}P_{H^n}}{\mathrm{d}Q_{H^n}}(h^n).$$

where $\forall Q_{H^n} \in \mathcal{Q}$, and $\mathcal{Q}$ the set of reference probability measures (**computable**) such that $P_{H^n} \ll Q_{H^n}$.

Information density lower bound

Theorem (Information density lower bound)

Let $Q_{H^n} \in \mathcal{Q}$ be a reference measure. Fix $r > 1$, and let $s \triangleq \frac{r}{r-1}$ denote its Hölder conjugate. Then, given a pair $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$,

$$\begin{aligned} \imath(x^n; y^n) \geq \log f_{Y^n|X^n}(y^n|x^n) - \frac{r-1}{r} D_r(P_{H^n} \| Q_{H^n}) \\ - \frac{1}{s} \log \mathbb{E}_{Q_{H^n}} \left[f_{Y^n|H^n}(y^n|H^n)^s \right], \end{aligned} \tag{5}$$

where $D_r(P_{H^n} \| Q_{H^n})$ is the Rényi divergence of order r.

Information density lower bound

Theorem (Information density lower bound)

Let $Q_{H^n} \in \mathcal{Q}$ be a reference measure. Fix $r > 1$, and let $s \triangleq \frac{r}{r-1}$ denote its Hölder conjugate. Then, given a pair $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$,

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where $D_r(P_{H^n} \| Q_{H^n})$ is the Rényi divergence of order r.

Operational meaning

Whenever the lower bound crosses the threshold, the actual information density must cross, too. The VLSF decoding rule holds with reliability guarantee.

Upper bound

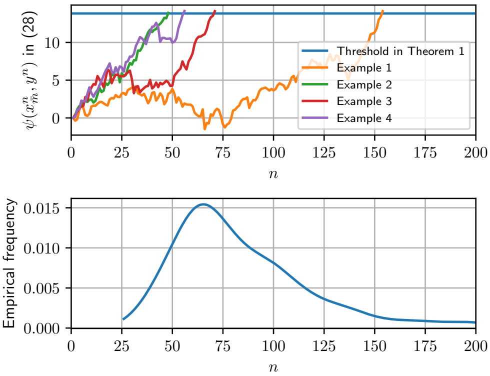
Theorem (Information density upper bound)

Let $Q_{H^n} \in \mathcal{Q}$*, and the likelihood ratio* $L(h^n)$*. Then, for every realization* (x^n, y^n)*,*

$$\imath(x^n; y^n) \leq \log f_{Y^n|X^n}(y^n|x^n) + D(P_{H^n} \| Q_{H^n}) - \mathbb{E}_{Q_{H^n}}\left[\log f_{Y^n|H^n}(y^n|H^n)\right], \tag{6}$$

where $D(P_{H^n} \| Q_{H^n})$ *denotes the Kullback-Leibler (KL) divergence.*

Performance characterization



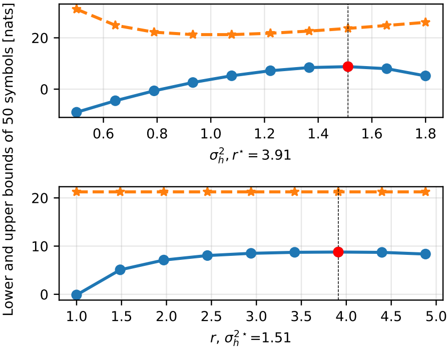
8

Specialization

- Q_{H^n} i.i.d. Gaussian auxiliary distribution
- P_X Gaussian signaling

[8] Chosen for mathematical tractability rather than strict optimality

Upper- and lower bound



- Non-negligible gap.

Conclusion

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- We develop a gradient-based framework to compute achievability bounds for sparse VLSF codes, leveraging the analyticity of saddlepoint approximations.
- The framework enables alternative decoding rules and reveals rate gains.
- Furthermore, performance characterization of VLSF codes could be extended to correlated noncoherent fading channels.

References

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Backup Slides

Saddlepoint Approximations

- Let $Z_i = \imath(X_{1,i}; Y_i)$ be information density per channel use.
- Denote $K_Z(s)$ the cumulant generating function (CGF) of Z.
- The CGF of $S_n = \imath(X_1^n, Y^n) = \sum_{i=1}^n Z_i$ is $K(s) = nK_Z(s)$.
- If Z has a density, the Lugannani-Rice approximation [1] gives, for each $n \in \mathbb{N}_{>0}$ the CDF[9] of

$$\mathbb{P}[S_n < \gamma] \approx \begin{cases} \Phi(\hat{w}) + \phi(\hat{w})(1/\hat{w} - 1/\hat{u}) & \gamma \neq \mathbb{E}[S_n], \\ \frac{1}{2} + \frac{K'''(0)}{6\sqrt{2\pi}K''(0)^{3/2}} & \gamma = \mathbb{E}[S_n], \end{cases}$$

where $\hat{w} = \mathrm{sgn}(\hat{s})\sqrt{2(\hat{s}\gamma - K(\hat{s}))}$ and $\hat{u} = \hat{s}\sqrt{K''(\hat{s})}$.

Takeaway:

$\mathbb{P}[\imath(X_1^{n_j}, Y^{n_j}) < \gamma]$ admits a closed-form expression in terms of the saddlepoint $\hat{s}$.

[9] The functions ϕ and Φ denote the standard normal probability density and CDF, respectively, and sgn(.) is the sign function.

Saddlepoint Approximations

For discrete Z, let k denote the smallest attainable lattice point above γ. Then, $\mathbb{P}[S_n < \gamma] \approx$

$$\begin{cases} \Phi(\hat{w}) + \phi(\hat{w})(1/\hat{w} - 1/\bar{u}) & k \neq \mathbb{E}[S_n], \\ \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{K'''(0)}{6K''(0)^{3/2}} - \frac{1}{2\sqrt{K''(0)}}\right) & k = \mathbb{E}[S_n], \end{cases}$$

where $\hat{w}$ is defined as in the continuous case, and the curvature correction term is given by $\bar{u} = (1 - e^{-\hat{s}})\sqrt{K''(\hat{s})}$.

Takeaway:

$\mathbb{P}[\imath(X_1^{n_j}, Y^{n_j}) < \gamma]$ admits a closed-form expression in terms of the saddlepoint $\hat{s}$.

Common Memoryless i.i.d. Channels

Additive White Gaussian Noise (AWGN) Channel:
$Y = X + N$, where $X \sim \mathcal{N}(0, p_0)$ and $N \sim \mathcal{N}(0, 1)$

$$Z = \frac{1}{2}\log(1+p_0) + \frac{1}{2}\left(\frac{Y^2}{p_0+1} - (Y-X)^2\right).$$

Centering the random variable Z as $\tilde{Z} = Z - \frac{1}{2}\log(1+p_0)$ The threshold $\tilde{\gamma} = \gamma - \frac{n}{2}\log(1+p_0)$ The CGF of $\tilde{S}_n$ is given by

$$K_{\tilde{S}_n}(s) = -\frac{n}{2}\log\left(1 - \frac{p_0 s^2}{p_0+1}\right),$$

and the corresponding saddlepoint $\hat{s}$ is given by

$$\hat{s} = \frac{-n + \sqrt{n^2 + 4\tilde{\gamma}^2(p_0+1)/(p_0)}}{2\tilde{\gamma}}.$$

Takeaway:

Saddlepoint $\hat{s}$ is in closed form with respect to γ and n for common channels.

Common Memoryless i.i.d. Channels

Binary Symmetric Channel (BSC): with crossover probability $\delta \in (0, \frac{1}{2})$ and input $X \sim \text{Bernoulli}(\frac{1}{2})$.

$$Z = \begin{cases} \log 2(1-\delta) & \text{w.p. } 1-\delta, \\ \log 2\delta & \text{w.p. } \delta, \end{cases}$$

The CGF of $\tilde{S}_n$ is $K_{\tilde{S}_n}(s) = n \log\left(\delta + (1-\delta) e^{\log \frac{1-\delta}{\delta} s}\right)$. Solving $K'_{\tilde{S}_n}(s) = k - n\log 2\delta$ gives a closed form expression for the saddlepoint:

$$\hat{s} = \frac{\log\left(\frac{(k/n - \log 2\delta)\delta}{(\log(2(1-\delta)) - k/n)(1-\delta)}\right)}{\log\left(\frac{1-\delta}{\delta}\right)}.$$

Binary Erasure Channel (BEC): Similar derivations.

Takeaway:

Saddlepoint $\hat{s}$ is in closed form with respect to γ and n for common channels.