

# Channel Capacity under the Subtractive Dithered Quantization Model

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**Abstract**—We study the capacity of an additive white Gaussian noise (AWGN) channel followed by a subtractive dithered uniform quantizer. Under the Schuchman conditions and with negligible overload probability, the system admits an additive-noise representation in which the effective noise is the sum of Gaussian and uniform components.

Capacity bounds are derived for this model when inputs are subject to an average-power constraint as well as a peak-amplitude constraint, where the latter accounts for the limited quantizer dynamic range. Specifically, a computable lower bound is obtained based on the entropy power inequality (EPI), using the maximum-entropy input under the above constraints. Tighter numerical lower bounds are derived using discrete input constellations with finite mass points. Finally, an upper bound is obtained by exploiting the fact that Gaussian distributions maximize entropy under a variance constraint.

Numerical results show that, for a  $K$ -level quantizer, discrete constellations with  $K$  mass points already achieve near-optimal rates among the tested families. Moreover, our upper bound is close to the lower bounds in the moderate-SNR regime; it thus represents a good and simple capacity approximation in this regime.

**Index Terms**—Channel capacity, dithered quantizer, analog to digital converter (ADC), amplitude-limited inputs.

## I. INTRODUCTION

Modern communication systems rely on digital signal processing, where ADCs convert the received analog waveform into a finite-resolution digital signal from which the transmitted message is reconstructed and decoded. Low-precision ADCs, operating with 1–3 bits per sample, have attracted significant research interest due to their potential to reduce power consumption and hardware complexity [1], [2], particularly in millimeter-wave (mmWave) and massive MIMO systems [3].

Understanding the fundamental information-theoretic limits imposed by coarse quantization on communication channel capacity is therefore a problem of both theoretical and practical importance. Closed-form channel capacity expressions are generally intractable, and the literature has focused on two main directions. One line of research studies analytically tractable cases, most notably the one-bit ADC model. For an AWGN channel followed by a sign quantizer, Singh, Dabeer, and Madhow showed that antipodal signaling achieves capacity under an average power constraint [4]. Another line

of work derives computable bounds for more general quantization models. For example, [5] analyzes linear transceivers with quantized DACs and ADCs by approximating quantization as additive Gaussian noise in the large-system limit, and derives both upper and lower bounds on capacity.

Smith [6] established that, for an AWGN channel under simultaneous amplitude and average power constraints on the channel input, the capacity-achieving input distribution is discrete with a finite number of mass points. The corresponding optimality conditions characterize both the support and the probability masses. Extensions of this result to broader channel models suggest that discreteness persists under fairly general conditions on the noise distribution [7]–[9].

For channels with quantized outputs, Singh et al. [4] further showed that, under both peak and average power constraints, the capacity-achieving input distribution is discrete with finite support. In particular, for a quantizer with  $K$  output levels, a symmetric input with at most  $K + 1$  mass points suffices.

In this paper, we adopt a complementary model based on *subtractive dithered quantization*. By adding an independent dither prior to quantization and subtracting it afterward, the input-output relation of the channel simplifies and can be expressed as  $Y = X + \bar{W}$ , where  $\bar{W}$  is the sum of Gaussian channel noise and independent uniform quantization noise. This representation holds under the Schuchman conditions [10], [11], which require that the dither be independent of the input and properly distributed, and that the quantizer must operate without overload, i.e., within its linear operating regime. Under these conditions, the quantization error becomes independent of the input and uniformly distributed.

To ensure that the quantizer operates in its linear regime  $[-\gamma, \gamma]$  with high probability, we impose a *peak amplitude constraint*  $|X| \leq A$ , so that the probability of overload is at most  $\varepsilon$ , for a small constant  $\varepsilon > 0$ .

Under this framework, we derive bounds on the capacity. Our upper bound is based on the variance of the output signal and the maximum-entropy property of the Gaussian distribution under a fixed second moment. We present the following lower bounds:

- an entropy power inequality (EPI) based bound, using the maximum-entropy input under both a peak amplitude and

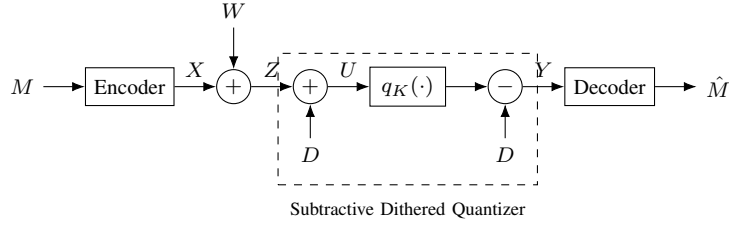


Fig. 1. Block diagram of the communication system model with subtractive dithered quantization at the receiver.

an average power constraint on the inputs; and

- discrete-input lower bounds obtained by numerically optimizing symmetric constellations with  $N = 2, 3, 4, 5, 8,$  and  $9$  mass points.

Numerical results confirm that the proposed upper and lower bounds are tight at moderate signal-to-noise ratios (SNRs). The EPI lower bound and the variance-based upper bound thus provide simple and tight approximations of capacity in this regime. Across all SNR regimes, our bounds suggest that when using  $K$ -level quantizers,  $N = K$  discrete mass points suffice to achieve a rate close to capacity; further increasing the number of mass points hardly improves the achievable rates. At low SNR, two-point constellations seem sufficient.

Note that the effective noise in our channel (consisting of Gaussian and uniform noises) has a continuous density with finite variance and satisfies regularity conditions as stated in [6]–[8]. These results thus prove that the capacity-achieving input at all SNR levels is indeed discrete and with finite support.

## II. SYSTEM MODEL

We consider a real, discrete-time AWGN channel

$$Z = X + W, \quad (1)$$

where  $W \sim \mathcal{N}(0, \sigma^2)$  is independent of  $X$ .

The receiver employs a  $K$ -level subtractive dithered uniform quantizer with dynamic range  $[-\gamma, \gamma]$  and step size

$$\Delta = \frac{2\gamma}{K}. \quad (2)$$

Specifically, an independent dither  $D \sim \mathcal{U}(-\Delta/2, \Delta/2)$  is added prior to quantization and subtracted afterward.

As shown in Fig. 1, the interval  $[-\gamma, \gamma]$  is partitioned into  $K$  uniform subintervals of width  $\Delta$ , whose midpoints define the reconstruction levels, with saturation at  $\pm(\gamma - \Delta/2)$  outside the range.

Under the Schuchman conditions for subtractive dithering [10], [11], and assuming that the probability of overload outside  $[-\gamma, \gamma]$  is sufficiently small, the overall input–output relation can be modeled as

$$Y = X + \bar{W}, \quad (3)$$

where the effective noise is

$$\bar{W} = W + W_q, \quad (4)$$

with  $W_q \sim \mathcal{U}(-\Delta/2, \Delta/2)$  independent of both  $X$  and  $W$ . The PDF of  $\bar{W}$ , obtained as the convolution of a Gaussian and a uniform density, along with the corresponding CDF, denoted  $f_{\bar{W}}(\cdot)$  and  $F_{\bar{W}}(\cdot)$ , are provided in Appendix A.

### A. Input Constraints

Let  $U = X + W + D$  denote the quantizer input. The restriction of the overload probability to be small directly limits the admissible channel inputs. We therefore impose the pointwise overload constraint

$$\Pr(|U| > \gamma \mid X = x) \leq \varepsilon, \quad \forall x \in \text{supp}(P_X), \quad (5)$$

for a given tolerance  $\varepsilon > 0$ . The overload probability is

$$\Pr(|U| > \gamma \mid X = x) = 1 - [F_{\bar{W}}(\gamma - x) - F_{\bar{W}}(-\gamma - x)], \quad (6)$$

which is non-decreasing in  $|x|$  by the symmetry and unimodality of  $f_{\bar{W}}$ . Consequently, the supremum over the support is attained at the boundary  $x = A$ . It therefore suffices to enforce the constraint at  $x = A$ , which results in the implicit equation

$$1 - [F_{\bar{W}}(\gamma - A) - F_{\bar{W}}(-\gamma - A)] = \varepsilon. \quad (7)$$

These constraints define the feasible input set: we study distributions  $P_X$  satisfying the *peak amplitude constraint*

$$|X| \leq A \quad (8)$$

and the *average power constraint*

$$\mathbb{E}[X^2] \leq P_0. \quad (9)$$

*Remark 1.* A strictly positive solution  $A > 0$  to (7) exists if and only if

$$\varepsilon > \varepsilon_t, \quad \varepsilon_t \triangleq 2(1 - F_{\bar{W}}(\gamma)),$$

which follows from evaluating the overload probability at  $A = 0$  and using monotonicity in  $A$ .

### B. Problem Formulation

We study the channel capacity under the input constraints derived above. We denote the capacity under the constraints in (8) and (9) by  $C(A, P_0)$ , defined as

$$C(A, P_0) = \sup_{P_X \in \mathcal{F}} I(X; Y), \quad (10)$$

where the feasible set is given by

$$\mathcal{F} = \{P_X : \mathbb{E}[X^2] \leq P_0, |X| \leq A\}, \quad (11)$$

and  $I(X;Y)$  denotes the mutual information between the channel input and output.

This class of capacity problems for additive-noise channels under input constraints has been studied extensively (see, e.g., [6], [8]). However, explicit characterizations are generally unavailable for non-Gaussian noise distributions such as  $\bar{W}$ , as well as for amplitude constraints induced by quantizer dynamics.

Under the additive noise model in (3), the mutual information can be expressed as

$$I(X;Y) = h(Y) - h(\bar{W}), \quad (12)$$

where  $h(\cdot)$  denotes differential entropy.

By symmetry of the noise and the constraint set, a capacity-achieving input can be chosen symmetric, which can be proved analogously to [4, Lemma 2]. Therefore, we restrict attention to symmetric inputs.

### III. THEORETICAL RESULTS

Finding a closed-form expression for the capacity is intractable. We derive upper and lower bounds on  $C(A, P_0)$ .

#### A. A Variance-Based Upper Bound on Channel Capacity

Since for a given variance, the entropy is maximized by a Gaussian distribution, we have the following upper bound on the capacity directly obtained from (12).

**Theorem 1** (Variance-Based Upper Bound). *The capacity of our model is upper bounded as*

$$C \leq \frac{1}{2} \log \left( 2\pi e \left( \min\{A^2, P_0\} + \sigma^2 + \frac{\Delta^2}{12} \right) \right) - h(\bar{W}), \quad (13)$$

where  $h(\bar{W}) := -\int_{-\infty}^{\infty} f_{\bar{W}}(w) \log f_{\bar{W}}(w) dw$  is the noise entropy, evaluated numerically.

In the low-power regime  $P_0 \rightarrow 0$ , the upper bound converges to  $\frac{1}{2} \log \left( 2\pi e \left( \sigma^2 + \frac{\Delta^2}{12} \right) \right) - h(\bar{W})$ , which is strictly positive; hence, the bound is loose in this regime.

As  $P_0 \rightarrow \infty$ , the upper bound saturates at  $\frac{1}{2} \log \left( 2\pi e \left( A^2 + \sigma^2 + \frac{\Delta^2}{12} \right) \right) - h(\bar{W})$ , proving that the capacity also saturates at a finite value. However, as our numerical results in the next section suggest, the saturation level of the upper bound exceeds that of the true capacity.

In contrast, in the moderate-power regime, the upper bound is observed to be close to the actual capacity and provides a simple estimate in this range.

#### B. Lower Bounds on Channel Capacity

1) *EPI Lower Bound*: The EPI yields the following bound.

**Theorem 2** (EPI-Based Lower Bound). *The capacity of our model is lower bounded as*

$$C(A, P_0) \geq \frac{1}{2} \log \left( e^{2h_{\max}(A, P_0)} + e^{2h(\bar{W})} \right) - h(\bar{W}), \quad (14)$$

where

$$h_{\max}(A, P_0) := \sup_{P_X \in \mathcal{F}} h(X). \quad (15)$$

The maximization in (15) leads to the following expression:

$$h_{\max}(A, P_0) = \begin{cases} \log(2A), & P_0 \geq \frac{A^2}{3}, \\ -\log c + \mu^* P_0, & P_0 < \frac{A^2}{3}, \end{cases} \quad (16)$$

with

$$c = \frac{\sqrt{\mu^*}}{\sqrt{\pi} \left( 1 - 2Q(\sqrt{2\mu^*} A) \right)}, \quad (17)$$

where  $\mu^* > 0$  solves

$$\frac{1}{2\mu} - \frac{Ae^{-\mu A^2}}{\sqrt{\pi\mu} \left( 1 - 2Q(\sqrt{2\mu} A) \right)} = P_0. \quad (18)$$

*Proof.* See Appendix B.  $\square$

From the above, at high power  $P_0 \geq A^2/3$ , the EPI-based lower bound saturates. The upper bound also saturates, but at a higher level than that of the EPI bound, as confirmed by the numerical results in the next section.

In the low-SNR regime, we have  $\mu^* \sim 1/(2P_0)$  for  $P_0 \rightarrow 0$ , and thus  $\mu^* P_0 \approx \frac{1}{2}$  and  $c \approx \sqrt{\frac{1}{2P_0}}$ , also because  $\mu^* \rightarrow \infty$  as  $P_0 \rightarrow 0$ . Substituting these approximations into (16) and (14) gives the low-SNR approximation

$$C(A, P_0) \gtrsim \frac{1}{2} \log \left( 1 + 2\pi e P_0 e^{-2h(\bar{W})} \right). \quad (19)$$

This lower bound decays to 0 linearly in  $P_0$  as desired (as  $\log(1+x) \approx x$  for small  $x$ ), however our numerical results suggest a constant offset (in dB) from capacity in the low-SNR regime, see the top of Figure 2.

In the moderate-SNR regime, our numerical results show that the EPI lower bound is close to our upper bound and thus provides a simple approximation of capacity.

2) *Discrete-Input Lower Bounds*: We obtain improved lower bounds by restricting the input distribution to symmetric finite-support constellations with  $N$  points. For  $N = 2$ , we use  $\mathcal{X} = \{-a, a\}$ . For  $N \geq 3$ , we consider

$$\mathcal{X} = \begin{cases} \{0, \pm a\beta_1, \dots, \pm a\beta_{\frac{N-3}{2}}, \pm a\}, & N \text{ odd}, \\ \{\pm a\beta_1, \dots, \pm a\beta_{\frac{N-2}{2}}, \pm a\}, & N \text{ even}, \end{cases} \quad (20)$$

where  $0 \leq \beta_1 \leq \dots \leq 1$ ,  $0 \leq a \leq A$ , and the probabilities are symmetric.

For any such constellation, with support points  $\{x_n\}_{n=1}^N$  and probabilities  $\{p_n\}_{n=1}^N$ , the power constraint becomes

$$\mathbb{E}[X^2] = a^2 \Psi(\boldsymbol{\beta}, \mathbf{p}), \quad \Psi(\boldsymbol{\beta}, \mathbf{p}) := \sum_{n=1}^N p_n \left( \frac{x_n}{a} \right)^2. \quad (21)$$

Hence, for fixed  $(\boldsymbol{\beta}, \mathbf{p})$ , the admissible amplitude satisfies

$$a \leq \min \left\{ \sqrt{\frac{P_0}{\Psi(\boldsymbol{\beta}, \mathbf{p})}}, A \right\}. \quad (22)$$

We evaluate 2-, 3-, 4-, 5-, 8-, and 9-point constellations as special cases of (20). For each family,  $I(X;Y)$  is maximized numerically over  $(\boldsymbol{\beta}, \mathbf{p}, a)$  subject to the peak-amplitude and

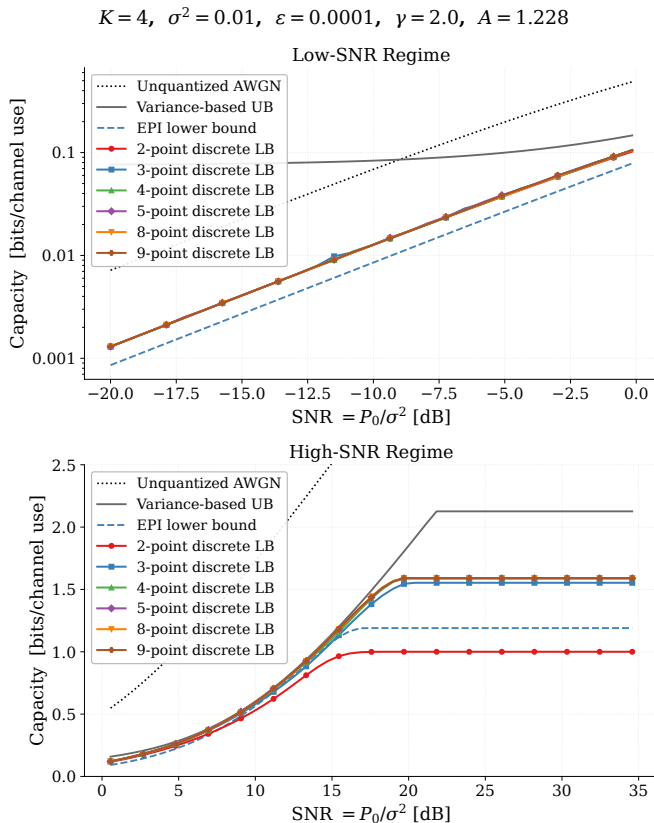


Fig. 2. Capacity bounds for  $K = 4$  quantization levels. *Top*: high-SNR regime. *Bottom*: low-SNR regime in logarithmic scale.

average-power constraints, leading to a sequence of increasingly tight lower bounds on the capacity  $C(A, P_0)$ .

*Remark 2.* The effective noise  $\bar{W}$  has a continuous density with finite variance and admits an analytic extension. Under such regularity conditions, the capacity-achieving input distribution is discrete with finite support; see classical results for amplitude-constrained channels [6]–[8].

#### IV. NUMERICAL RESULTS

We evaluate the bounds numerically for  $\sigma^2 = 10^{-2}$ ,  $\gamma = 2$ ,  $\varepsilon = 10^{-4}$ , and different values of  $K$ , with  $\text{SNR} \triangleq P_0/\sigma^2$ . The amplitude constraint  $A$  is computed from these parameters via (7). For the above choice of  $\gamma$  and  $K \in \{2, 4, 8\}$ , the value of  $A$  does not vary significantly with  $\varepsilon$  and remains nearly constant over a wide range of  $\varepsilon$ , for example when  $\varepsilon$  is reduced to  $10^{-6}$ .

The choice  $\gamma = 2$  is motivated by practical considerations as explained in [1], [2]. To illustrate the theoretical impact of the dynamic range, we also present results for  $\gamma \in \{5, 10\}$ .

##### A. Capacity Bounds for $K = 4$

Fig. 2 shows the capacity bounds derived in the previous section for  $K = 4$ . For reference, we also plot the capacity of the unquantized Gaussian channel, i.e., without quantization noise  $W_q$  and without the peak amplitude constraint (8).

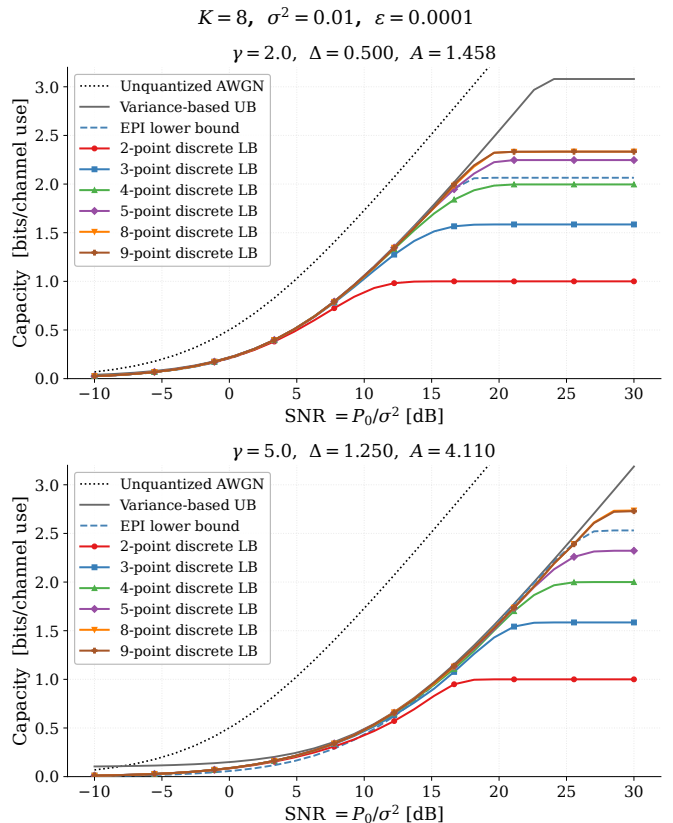


Fig. 3. Capacity bounds for  $K = 8$  quantization levels and two values of the ADC dynamic range:  $\gamma = 2$  (top) and  $\gamma = 5$  (bottom).

In the moderate- and high-SNR regimes, the numerical results confirm the analytical observations of Section III. Our variance-based upper bound significantly improves over the pure Gaussian upper bound, and seems tight in the moderate SNR regime; its high-SNR saturation level, however, lies considerably above all lower bounds.

The EPI lower bound is tight at moderate SNR but degrades relative to the discrete-input lower bounds at high SNR. Among the discrete-input lower bounds, performance increases with the number of mass points for  $N = 2, 3, 4$ ; beyond  $N = K = 4$ , the gain from adding further mass points is negligible.

At low SNR, the variance-based upper bound saturates at a strictly positive value and is therefore looser than the unquantized Gaussian capacity in this regime. A gap between the best upper and lower bounds persists throughout. In the low-SNR regime, the two-point lower bound performs best, and increasing  $N$  beyond 2 does not lead to any noticeable improvement. It also strictly outperforms the EPI lower bound.

##### B. Capacity Bounds for $K = 8$

Fig. 3 shows the capacity bounds for  $K = 8$  and  $\gamma \in \{2, 5\}$ . The behavior is generally similar to the case  $K = 4$ , with all bounds larger for  $K = 8$  under the same value of  $\gamma$ . As observed for  $K = 4$ , the 9-point constellation yields no noticeable improvement over the 8-point one.

$K = 4$ ,  $\gamma = 2.0$ ,  $\Delta = 1.000$ ,  $\sigma^2 = 0.01$ ,  $\varepsilon = 0.0001$ ,  $A = 1.228$

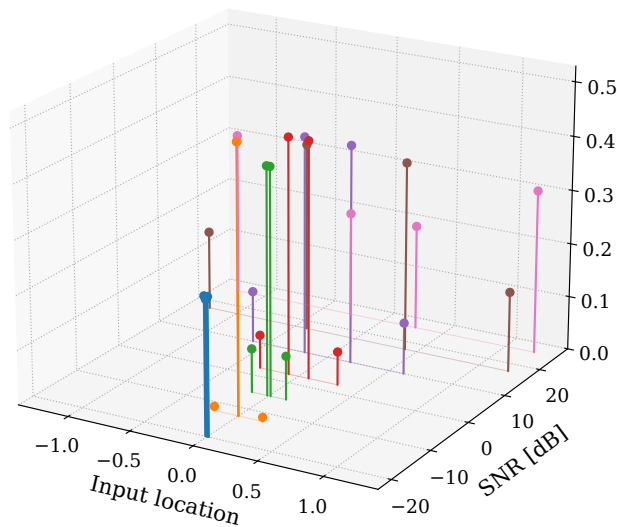


Fig. 4. The optimal 4-point input distribution, as a function of SNR. Each stem represents a mass point, with its height the corresponding probability.

Our bounds further allow us to conclude that larger values of  $\gamma$  reduce the capacity in the low-SNR regime but increase it in the high-SNR regime. This holds because a large value of  $\gamma$  results in a more detrimental uniform noise  $W_q$ , which dominates at low SNR, but also increases the amplitude constraint  $A$ , which becomes the limiting factor at high SNR.

In a similar sense, for the larger value  $\gamma = 5$ , each  $N$ -point lower bound approaches  $\log_2 N$  bits/channel use in the high-SNR regime. This holds because a large amplitude constraint  $A$  allows the  $N$  constellation points to be well separated, which results in a small conditional entropy  $H(X|Y)$ . Furthermore, at high SNR, the input distribution can be chosen nearly uniform over the constellation, so that  $I(X; Y) = H(X) - H(X|Y) \approx H(X) \approx \log_2(N)$ .

### C. The Maximizing Input Distribution as a Function of SNR

Fig. 4 shows the optimal 4-point input distribution as a function of SNR. As SNR increases, the distribution evolves from a Gaussian-like density with mass concentration near the origin to a more uniform allocation throughout the support  $|x| \leq A$ . This reflects the transition from a *noise-limited regime*, where Gaussian-like inputs are favorable, to a *constraint-limited regime*, where maximizing the input entropy under the constraint  $|X| \leq A$  becomes dominant.

### D. Comparison with [4] ( $K = 2$ )

Fig. 5 compares the achievable rate of our subtractive dithered quantizer model with  $K = 2$  quantization levels to the capacity derived in [4] for a one-bit sign quantizer without dither and without a peak amplitude constraint reflecting the finite dynamic range of the quantizer. For clarity, we plot only

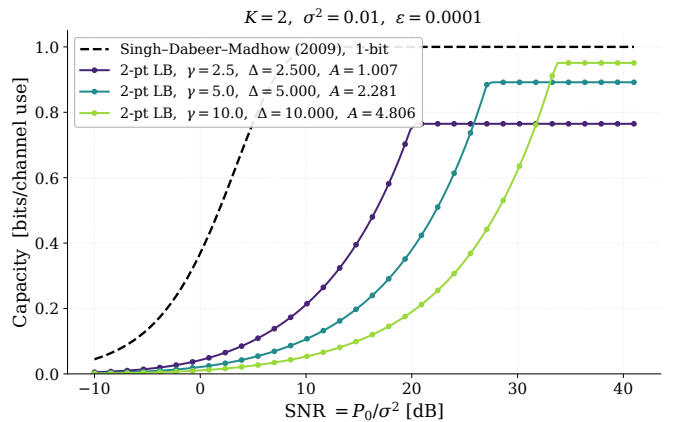


Fig. 5. Capacity of a one-bit sign quantizer under an average power constraint [4] compared to the 2-point discrete-input capacity lower bound for the  $K = 2$  subtractive dithered quantizer model with different  $\gamma$ -values.

the 2-point discrete-input lower bound for our model, which is expected to be close to capacity.

We observe a significant capacity gap between the two models at low SNR, which seems to stem from the noise introduced by the dither. At high SNR, the achievable rate of the subtractive dithered model approaches the capacity in [4] as  $\gamma$  increases, since a larger dynamic range relaxes the induced peak amplitude constraint. However, for small values of  $\gamma$ , the gap remains substantial and reflects the stringent peak amplitude constraint in our model, a practical limitation absent in [4].

## V. CONCLUSIONS

This paper studied the capacity of a real AWGN channel with a subtractive dithered  $K$ -level quantizer. By the dithering principle [10], [11], this setup is modeled as a channel with both uniform and Gaussian additive noises subject to joint peak amplitude and average power constraints. The peak amplitude constraint arises from the finite dynamic range of the ADC and is primarily determined by the number of quantization levels  $K$  and the quantizer range, along with other system parameters.

We derived an EPI-based lower bound on the capacity of your model using a truncated Gaussian input, and tighter numerical lower bounds using inputs with a finite number of mass points. A variance-based upper bound on capacity was also established.

Our numerical comparison of the above bounds shows that the bounds are tight in the moderate SNR regime, where the variance-based upper bound and the EPI lower bound thus provide simple approximate characterizations of capacity. At both low and high SNR, the EPI bound is strictly below the bounds obtained with a finite number of mass points, and the upper and lower bounds do not coincide. The numerical lower bounds are expected to be close to capacity. Future work will focus on finding improved capacity upper bounds, for example using the duality framework in [12], [13].

Another observation is that, for a system with  $K$  quantization bins, a discrete input constellation with  $N = K$  points suffices to approximate capacity closely. At low SNR, a 2-point constellation is sufficient, irrespective of the number of quantization levels  $K$ .

#### APPENDIX A

##### DERIVATION OF THE EFFECTIVE NOISE DISTRIBUTION

Let  $\bar{W} = W + W_q$ , where  $W \sim \mathcal{N}(0, \sigma^2)$  and  $W_q \sim \mathcal{U}(-\Delta/2, \Delta/2)$  are independent.

##### A. Probability Density Function

The PDF of  $\bar{W}$  is given by the convolution

$$f_{\bar{W}}(w) = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \frac{1}{\sigma} \phi\left(\frac{w-z}{\sigma}\right) dz, \quad (23)$$

which evaluates to

$$f_{\bar{W}}(w) = \frac{1}{\Delta} \left[ Q\left(\frac{w-\frac{\Delta}{2}}{\sigma}\right) - Q\left(\frac{w+\frac{\Delta}{2}}{\sigma}\right) \right]. \quad (24)$$

##### B. Cumulative Distribution Function

The CDF of  $\bar{W}$  is given by

$$F_{\bar{W}}(w) = 1 + \frac{1}{\Delta} \left[ \left( w - \frac{\Delta}{2} \right) Q\left(\frac{w-\frac{\Delta}{2}}{\sigma}\right) - \left( w + \frac{\Delta}{2} \right) Q\left(\frac{w+\frac{\Delta}{2}}{\sigma}\right) + \sigma \left( \phi\left(\frac{w+\frac{\Delta}{2}}{\sigma}\right) - \phi\left(\frac{w-\frac{\Delta}{2}}{\sigma}\right) \right) \right]. \quad (25)$$

#### APPENDIX B

##### DERIVATION OF THE EPI-BASED LOWER BOUND

By the entropy power inequality,

$$I(X; Y) = h(X + \bar{W}) - h(\bar{W}) \quad (26)$$

$$\geq \frac{1}{2} \log\left(e^{2h(X)} + e^{2h(\bar{W})}\right) - h(\bar{W}). \quad (27)$$

Maximizing over feasible inputs gives

$$C(A, P_0) \geq \frac{1}{2} \log\left(e^{2h_{\max}(A, P_0)} + e^{2h(\bar{W})}\right) - h(\bar{W}), \quad (28)$$

where

$$h_{\max}(A, P_0) = h(X^*). \quad (29)$$

According to [14, Ch. 11], the optimal density has the form

$$f_X^*(x) \propto e^{-\mu x^2}, \quad |x| \leq A, \quad (30)$$

for some parameter  $\mu \geq 0$ .

Two cases arise:

*a) Case 1:  $\mu = 0$  (inactive power constraint):* In this case, the distribution reduces to the uniform density over  $[-A, A]$ , and

$$h_{\max}(A, P_0) = \log(2A), \quad (31)$$

which is feasible when

$$P_0 \geq \frac{A^2}{3}. \quad (32)$$

*b) Case 2:  $\mu > 0$  (active power constraint):* In this case, the density is a truncated Gaussian of the form

$$f_X^*(x) = \frac{\sqrt{\mu}}{\sqrt{\pi}(1-2Q(\sqrt{2\mu}A))} e^{-\mu x^2}, \quad |x| \leq A, \quad (33)$$

and the parameter  $\mu^* > 0$  is determined from the power constraint

$$\mathbb{E}[(X^*)^2] = P_0. \quad (34)$$

This leads to the equation

$$\frac{1}{2\mu} - \frac{Ae^{-\mu A^2}}{\sqrt{\pi\mu}(1-2Q(\sqrt{2\mu}A))} = P_0, \quad (35)$$

which uniquely determines  $\mu^*$ .

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